A Mathematical Description of Layered Manufacturing Fabrication

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Abstract
This study is attempted to use a mathematical definition to describe the principles of Layer Manufacturing Processing. The concept of model decomposition (layered subtraction for 3D model slicing) and material accumulation (layered addition for prototyping fabrication) and the associated sequence function and sequence potential to explain and define the layered manufacturing processing is presented. In the mathematical description, a 3D CAD model is graphically represented by a set of points collected within the bounded surfaces. In addition to its geometric feature, a processing sequence indicator is also assigned to each point as an attribute to associate with its slicing and fabricating sequence. Model decomposition slices the collected points into a series of point sets according to their sequence indicator, and material accumulation processes the layered fabrication by stacking the point sets to form the designed object. A scalar field function is used to express the variation of the sequence indicators for the selected point sets and to define iso-sequence planes. The iso-sequence planes are the processing layers consisting of all points with the same sequence indicator. Material accumulation is conducted in the gradient direction of each iso-sequence plane. Example of using proposed scalar field function and the iso-sequence plane for flat and no-flat layered prototyping processing is also presented.

Keywords: Layered Manufacturing, Rapid Prototyping, Solid Freeform Fabrication

1. Introduction

Layered Manufacturing, or frequently referred as Rapid Prototyping and Manufacturing or Solid Freeform Fabrication, has been proved as an effective tool for product development due to its advantages of shorten product development cycle, time to market and product quality. Different layered manufacturing techniques and systems have been developed and reported[1-7]. For example, Stereolithography Apparatus (SLA), the earliest layered manufacturing processing, solidifies the liquid resin on the selected region by an ultraviolet laser. When the processing layer is solidified, fresh resin is emerged to form a new layer for processing. 3D solid object is thus built layer by layer through this layered forming processing. Laminated Object Manufacturing (LOM) cuts sections from coated sheet paper and laminates them layer by layer to fabricate 3D solid part. Selective Laser Sintering (SLS) produces the sections by selective sintering the powder plastic, wax, ceramics, or metal through a CO_2 laser beam. In Fuse Deposition Modeling (FDM), material is melted and extruded from a thin nozzle. With the controlled nozzle's movement, the fine molten material is paved in the section layer and fuse together with other layers to form a 3D solid object.

2. The Decomposition-Accumulation Principle of Layered Manufacturing

We divide the processes of layered manufacturing into following two procedures: Model Decomposition and Material Accumulation. Figure 1 presents a flow chart to describe these two procedures.

In the model decomposition processing, a 3D model is sliced into a series of sections. Those sections are the processing layers for later material accumulation. The processing paths are then generated according to the specific requirement of freeform techniques, such as filling or offsetting. In the material accumulation processing, fabrication tools move along the defined processing paths to add the materials and stack or bind them to the previous layers. The model decomposition slices the CAD model or a continuous volumetric object into the discontinuously discrete layers and generates processing path for the accumulation process, while the material accumulation stacks the discontinuously discrete layers to produce physical prototypes. The principle of layered manufacturing is based on these two procedures. Figure 2 describes a hierarchical structure of the decomposition-accumulation procedures. According to Figure 2, a 3D volume is decomposed along one of the three directional axes to form series discrete surfaces, lines and points. Those discrete entities (surfaces, lines and points) are then stacked and transformed into the physical entities through the material accumulation. The model decomposition procedure has three levels of sub-decompositions: body sub-decomposition, surface sub-decomposition, and line sub-decomposition. The body sub-decomposition slices a 3D volume into 2D sections. The body sub-decomposition is necessary to all kinds of current layered manufacturing processing.

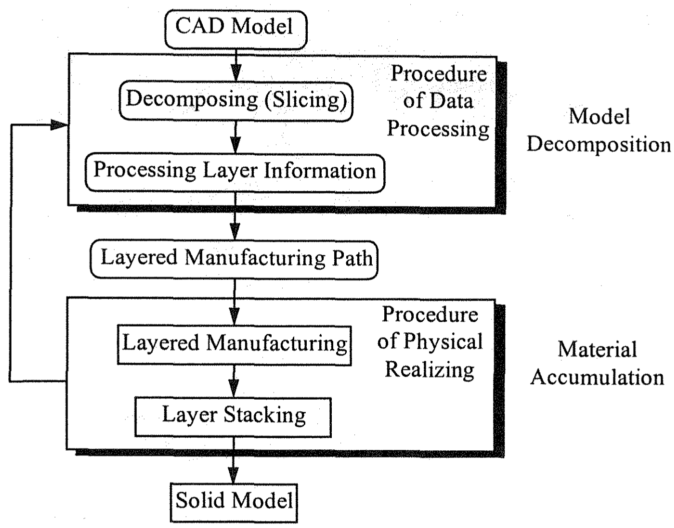

Figure 1 Layered manufacturing processing

After the body sub-decomposition, some layered manufacturing systems require further decomposition to define the processing paths. This is the second level decomposition and is called as the surface sub-decomposition. The third level decomposition is to decompose the section lines into section points. We define this as the line sub-decomposition. It should be pointed that the definitions of section lines and section points are abstract and only serve for the explanation purpose. Processing methods used in the layered manufacturing may be different from one system to another. This will result in different algorithms and process paths for the surface decomposition as well as the line decomposition. However, the principle of the model decomposition and its sub-decomposition for body, surface and line are all alike. For example, Laminate Object Manufacturing only requires the body sub-decomposition while Model Maker of Sanders requires all three level sub-decompositions.

Limitations of material thickness and fabrication efficiency constraints that the processing layers can not be too thin and the resolution density can not be too high during the body sub-decomposition. Usually, the interval of each body sub-decomposition is generally between 0.1mm to 0.25mm and the resolution from 4/mm to 10/mm. This is much lower than that in surface sub-decomposition and line sub-decomposition which the resolution densities are limited only by the material particulate dimension and resolution ratio of the control system movement. For most layered manufacturing systems, the material particulate dimension and the resolution ratio of the control system movement could be in the range of 0.01mm, and their resolutions could be greater than 100/mm. Therefore, out of three level of sub-decompositions, the body sub-decomposition will produce the most error than the other two. For example, the most common staircase appearance along the prototyping surface is due to the error of the body sub-decomposition. Since the body sub-decomposition is an essential processing for model decomposition in the layered manufacturing, this study is to focus on this discrete process and trying to use a mathematical description to define and to explain the layer manufacturing processing.

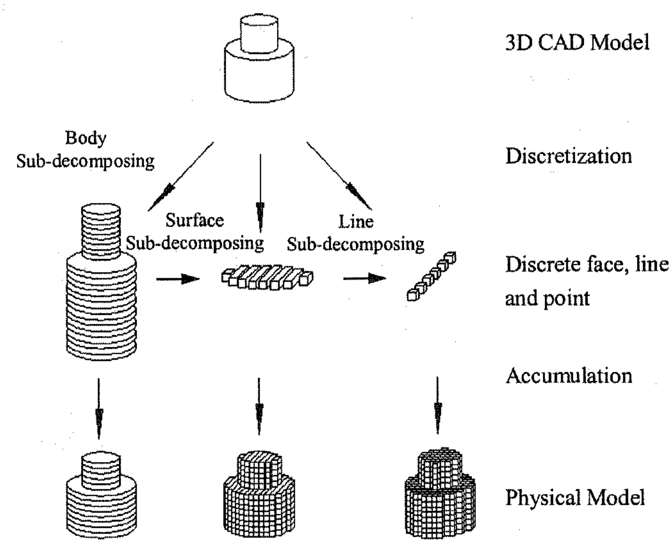

Figure 2. Hierarchy of decomposition-accumulation procedure

3. A Mathematics Description of Layer Manufacturing Processing

3.1 The representation of a 3D shape

In this paper, a 3D volume is represented as a collection of points within the bounded surface defined as: a 3D shape "V", with an entire exterior closed surface "S", is the combination of all points within$S$, "P_i" and on S, "P_b":

$$V = P_i \cup P_b \tag{1}$$

where, S is assumed to be a continuous, closed and self-non-intersecting surface. For a partitioned surface, S is the union of the partitioned areas "s_i".

$$S = s_1 \cup s_2 \cup s_3 \cup \cdots \cup s_i \tag{2}$$

where, i = 1, 2, 3, ... m, represents the number of the partitioned areas.

For example, the *stl* format, a *de facto* standard graphical interface of layered manufacturing technology, represents the part's surface by a set of triangular facets[6]. In this case, s_i corresponds to the triangular facet, which is the content in *stl* file, and S corresponds to the join of all the facets, which is the whole surface of part needs to be fabricated.

3.2 Sequence and Sequence Function

In layered manufacturing process, a 3D object is fabricated by decomposing and stacking sequence. Layers, the general forming unit in layered manufacturing process, can be considered as the material group with a certain sequence number. Once the layered manufacturing process is selected and the fabricating orientation is defined, the sequence, which is the order of the material accumulation, is setup and applied onto the 3D shape. Each point belongs to the 3D shape will be assigned with a sequence number to indicate the layer they should belong to. The 3D shape is then separated into layers according to the sequence applied to it. This separating operation is termed as decomposition processing, while fabricating part by stacking the discrete layers one by one following the sequence is termed as the material accumulation. The decomposition and accumulation should follow the same sequence otherwise the part's shape may be altered from the original CAD model. Fabricating by a specified sequence is another feature for layered manufacturing process.

We consider that the sequence is a number dispersal in 3 dimensional space $\mathscr{V}$ in which the process of layered manufacturing is realized. The sequence function is assumed as a scalar function $\varphi(x,y,z)$ defined in $\mathscr{V}$. To completely describe a point P in $\mathscr{V}$, we need not only to define its coordinate $p(x, y, z)$ for its spatial position, but also the value of sequence function $\varphi(x, y, z)$ to indicate the layer it belongs to and when for the point to be processed with the material accumulation.

The sequence function $\varphi(x, y, z) \in [a, b]$, if

$$\exists C = \{c_i; a \le c_0 < c_1 < \cdots < c_{i-1} < c_i < c_{i+1} < \cdots < c_n \le b, i = 1,2,\cdots,n\} \tag{3}$$

where, C is an ordered real number set and c_i represents the ith layer with n layers in total. a and b represents the values of the upper and lower bounds. For a volume defined in the ith layer:

$$v_i = V \cap P_i \tag{4}$$

where, V is the 3D shape, and P_i is the set of points whose sequences are all between c_{i-1} and c_i:

$$\begin{cases} P_i = \{p(x,y,z); c_{i-1} \le \varphi(x,y,z) < c_i\} & i = 1,2,\cdots,n-1 \\ P_n = \{p(x,y,z); c_{n-1} \le \varphi(x,y,z) \le c_n\} \end{cases} \tag{5}$$

and

$$v_i \cap v_j = \varnothing, \quad i \ne j, i,j = 1,2,\cdots,n \tag{6}$$

$$V = v_1 \cup v_2 \cup \cdots \cup v_n \tag{7}$$

The geometric description of the 3D shape V, its boundary S and the sequence function $\varphi(x, y, z)$ is presented in Figure 3. In this figure, the sub-volume v_i at the ith layer consists of all points whose sequences are between c_{i-1} and c_i within the 3D shape V.

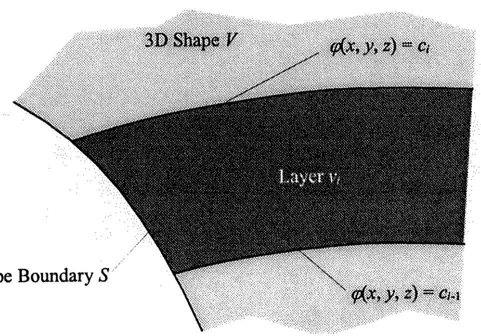

Figure 3 Geometric description of the processing layer

Current layered manufacturing techniques are difficulty to produce the exact boundary of S described in Figure 3. The boundary of layer is usually formed in straight with its obliquity normally parallel to the fabrication orientation. The actual boundary of the 3D prototyping part is staircase-like. Therefore, the processing layer v_i' by the layered manufacturing process is not equal to the geometrically defined sequence layer v_i, as shown in Figure 4.

The error e_i is the difference between the sequence layer v_i and the processing layer v_i':

$$e_i = (v_i - v_i') \cup (v_i' - v_i) \tag{8}$$

Because e_i is produced from the layered manufacturing processing, we call e_i the layered process error of ith layer. For an actual 3D prototyping part, the total layered process error E is:

$$E = e_1 \cup e_2 \cup \cdots \cup e_n \tag{9}$$

The layered process error E or e_i is related to the layer's geometric border obliquity and the processing orientation of the material accumulation. The detail analysis and quantity prediction of the layered process error was discussed in somewhere else[3].

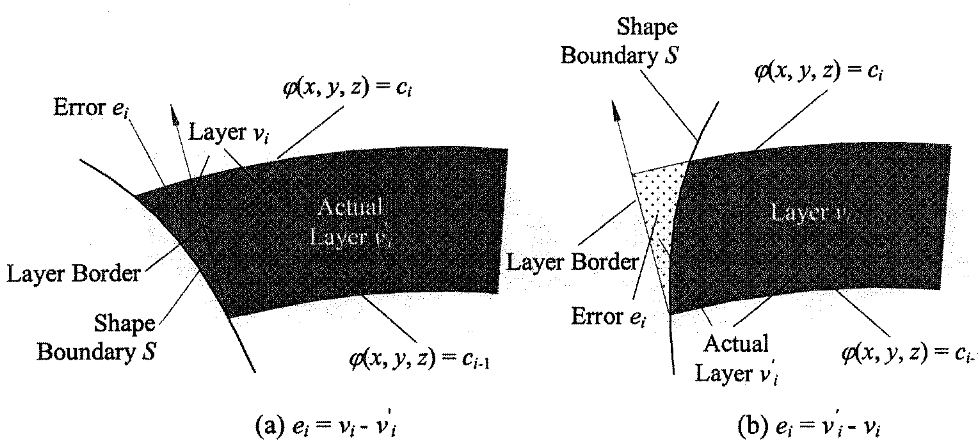

Figure 4. Error between the sequence layer and processing layer

3.3 Iso-sequence Plane and Sequence Potential

According to the field theory, the sequence function $\varphi(x, y, z)$ is a scalar function distributed in a 3D space $\mathcal{V}$. Let's assume a series of iso-sequence planes and $\varphi(x, y, z)$ be equal to the ordered set c_i as shown in Figure 5:

$$\varphi(x, y, z) = c_i, \quad i = 1, 2, \cdots n \tag{10}$$

The iso-sequence plane consists of all points with the same sequence indicator. In the decomposition processing, these iso-sequence planes are what we use to intersect and slice the designed 3D object. In the accumulation processing, these planes are where the layer is generated. For example, in SLA, the iso-sequence plane is the surface of the resin vat, on which the layer will be solidified from the liquid resin. In LOM, the iso-sequence plane is the top plane of the laminated part, on which new coated paper will be attached and cut into the sections. In SLS, it is the plane in which the fresh powder is spread and sintered. In current layered manufacturing processes, the iso-sequence plane is normally a 2D flat surface parallel to the XY plane.

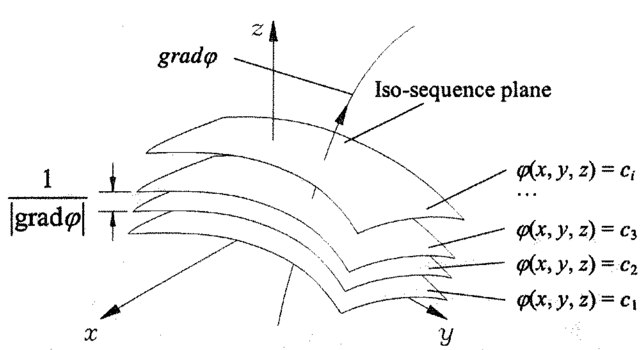

Figure 5. Iso-sequence plane and Sequence function

The gradient of the sequence function is:

$$\mathbf{d}(x,y,z) = grad\varphi = \frac{\partial\varphi}{\partial x}\mathbf{i} + \frac{\partial\varphi}{\partial y}\mathbf{j} + \frac{\partial\varphi}{\partial z}\mathbf{k} = X(x,y,z)\mathbf{i} + Y(x,y,z)\mathbf{j} + Z(x,y,z)\mathbf{k} \tag{11}$$

where **i**, **j** and **k** are the unit vectors in $\mathcal{V}$ along X, Y, Z axis, respectively. $\boldsymbol{d}(x, y, z)$ is spatial vector representing the gradient of the sequence function. Since $\boldsymbol{d}(x, y, z)$ is perpendicular to the iso-sequence plane everywhere, as shown in Figure 5, we can use the vector of $\boldsymbol{d}(x, y, z)$ to represent the processing orientation, i.e., along the direction of $\boldsymbol{d}(x, y, z)$, the 3D design model will be dispersed in the decomposition procedure and the material accumulation will be conducted in the material accumulation procedure. To this reason, we define $\boldsymbol{d}(x, y, z)$ as the sequence potential. Now each point in the space $\mathcal{V}$ will be associated with both sequence indicator $\varphi(x, y, z)$ and sequence potential $\boldsymbol{d}(x, y, z)$ defined from $\varphi(x, y, z)$.

The sequence function can also be derived from the sequence potential $\boldsymbol{d}(x, y, z)$ as following:

$$\begin{aligned}\varphi(x,y,z) = \varphi(x_0,y_0,z_0) + \int_{x_0}^{x} X(x,y_0,z_0)dx \\ + \int_{y_0}^{y} Y(x,y,z_0)dy + \int_{z_0}^{z} Z(x,y,z)dz\end{aligned} \tag{12}$$

where (x_0, y_0, z_0) is the coordinate of initial point and $\varphi(x_0, y_0, z_0)$ is the initial sequence number. As the requirement of iso-sequence plane defined by $\varphi(x, y, z)$, the vortices of $\boldsymbol{d}(x, y, z)$ must be equal to zero :

$$\frac{\partial X}{\partial y} = \frac{\partial Y}{\partial x}, \quad \frac{\partial Y}{\partial z} = \frac{\partial Z}{\partial y}, \quad \frac{\partial Z}{\partial x} = \frac{\partial X}{\partial z} \tag{13}$$

The module of sequence potential $|\boldsymbol{d}(x, y, z)|$ represents the maximum change rate of the sequence number $\varphi(x, y, z)$ at point (x, y, z). It indicates the density of the number of iso-sequence planes within a unit length. If we define the module $|\boldsymbol{d}(x, y, z)|$ equal to the reciprocal of the slicing interval, h_i, which is the distance between two adjacent iso-sequence planes:

$$|\boldsymbol{d}(x,y,z)| = \frac{1}{h_i} \tag{14}$$

then $|\boldsymbol{d}(x, y, z)|$ expresses the layer density which indicates how many layers should be sliced and fabricated within a unit length.

4. Application Examples

4.1 Description of Current Layered Manufacturing Processing

In the current layered manufacturing process, the fabricating direction is usually defined vertically upward and the layer thickness is uniform. Assume that the fabricating orientation is along Z-axis and the fabricating layer is parallel to the XY plane with a uniform thickness h, as shown in Figure 6, the sequence potential $\boldsymbol{d}_c$ is then defined as:

$$\boldsymbol{d}_c(x,y,z) = \frac{1}{h}\mathbf{k} \tag{15}$$

The corresponding sequence function $\varphi_c(x, y, z)$ is obtained as:

$$\varphi_c(x,y,z) = \varphi_c(x_0,y_0,z_0) + \int_{z_0}^{z} \frac{1}{h}dz = \varphi_c(x_0,y_0,z_0) + \frac{1}{h}(z - z_0) \tag{16}$$

where, the subscript c represents the current layered manufacturing processing, and $\varphi_c(x_0, y_0, z_0)$ represents the sequence indicator of the initial layer at z_0. In most cases, the sequence indicator always starts from zero.

$$\varphi_c(x_0, y_0, z_0) = 0 \tag{17}$$

z_0 is position of the initial layer. If the slicing starts from XY plan, then z_0 is equal to zero. Thus,

$$\varphi_c(x, y, z) = \frac{z}{h} \tag{18}$$

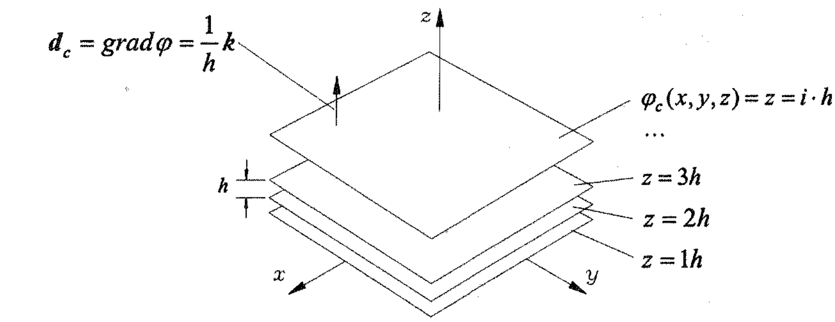

Figure 6. Sequence function and sequence potential for current LM processing

From the z coordinate we can identify which layer a given point should belong to. The iso-sequence planes for current layered manufacturing processes is derived from:

$$\begin{aligned}
\varphi_c(x, y, z) &= \frac{z}{h} = 1 &\Rightarrow\quad & z = h; \\
\varphi_c(x, y, z) &= 2 &\Rightarrow\quad & z = 2h; \\
\varphi_c(x, y, z) &= 3 &\Rightarrow\quad & z = 3h; \\
&\dots \\
\varphi_c(x, y, z) &= i &\Rightarrow\quad & z = i \cdot h.
\end{aligned}$$

where the ordered set C is $\{c_i;\ c_i = 0, 1, 2, \dots\}$ and $z = ih$ represents a series of flat planes parallel to each other and to the XY plane. Those flat planes are well suitable to be used to describe the current layered manufacturing processes because most of them can produce flat surfaces only.

4.2 Description of No-flat Layered Manufacturing

Because of the disadvantage of staircase produced by the flat layered manufacturing, no-flat layered manufacturing becomes a more realistic process for freeform fabrication. For a non-planar 3D object shown in Figure 7, the iso-sequence planes are defined as a set of curved surfaces.

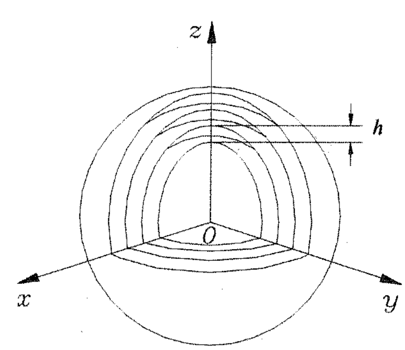

Figure 7. The concentric sphere iso-sequence planes

For instance, suppose a layered manufacturing process with a radial sequence potential, the material will then be added in the radial direction. The sequence potential can be obtained by:

$$\boldsymbol{d}_r = grad\varphi = \frac{1}{h\sqrt{x^2 + y^2 + z^2}}(x\mathbf{i} + y\mathbf{j} + z\mathbf{k}) \tag{19}$$

Let's assume that the vector module defined here be equal to 1/h. This means that we assume an isotropic uniform layer thickness, *h,* across the thickness direction. Let $\varphi(x_0, y_0, z_0)$ be equal to 0, and (x_0, y_0, z_0) equal to (0,0,0), from Equations 11 and 12, we derive the sequence function $\varphi_r(x, y, z)$ as:

$$\varphi_r(x,y,z) = \int_0^x \frac{1}{h\sqrt{x^2}}dx + \int_0^y \frac{1}{h\sqrt{x^2+y^2}}dy + \int_0^z \frac{1}{h\sqrt{x^2+y^2+z^2}}dz = \frac{1}{h}\sqrt{x^2+y^2+z^2} \tag{20}$$

then:

$$\varphi(x,y,z) = \frac{1}{h}\sqrt{x^2+y^2+z^2} = 1 \quad \Rightarrow \quad \sqrt{x^2+y^2+z^2} = h;$$

$$\varphi(x,y,z) = 2 \quad \Rightarrow \quad \sqrt{x^2+y^2+z^2} = 2h;$$

$$\ldots$$

$$\varphi(x,y,z) = i \quad \Rightarrow \quad \sqrt{x^2+y^2+z^2} = i\cdot h.$$

The iso-sequence planes, as showed in Figure 7, are a set of concentric sphere defined by $\sqrt{x^2+y^2+z^2} = i\cdot h$ with an interval h.

5. Conclusion

This study uses a mathematical description to explain the principle of layered manufacturing processing through the model decomposition: to decompose a 3D model into a series of sequence layers, and the material accumulation: to accumulate and stack processing layer into physical prototype. The 3D model is a set of points, which are bounded by the surface. Each point is associated with spatial position and sequence indicator. The sequence function $\varphi(x, y, z)$ and sequence potential $\boldsymbol{d}(x, y, z)$ are defined to correlate the iso-sequence planes which consist of the point sets with the same sequence indicator and the orientation of material accumulation. Example of using the derived mathematical formulation to describe the current layered manufacturing processing is also presented.

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