A minimum bounding box algorithm and its application to rapid prototyping

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ABSTRACT

This paper describes a method for determining the minimum bounding box of an arbitrary solid. The method simplifies the complex three-dimensional problem by projecting the solid onto the three principal planes and makes use of the projected contours for analysis. The orientations of the contours are determined by rotating them within a specific angle range. These orientations are then used to approximate the orientation of the solid so that its bounding box volume is minimised.
Keywords: bounding box, iterative method, minimum

INTRODUCTION

The problem of determining the minimum bounding box occurs in a variety of industrial applications likes packing and optimum layout design. The algorithm can be applied to many other fields, ranging from a straightforward consideration of whether an object will fit into a predetermined container, or whether it can be made from standard sized stock material[7], to uses in constructing bounding boxes in solid modellers to improve their performance. Freeman and Shapira[3] described a method for finding the rectangle of minimum area in which a given arbitrary plane curve can be contained. They proved that the rectangle of minimum area enclosing a convex polygon has a side collinear with one of the edges of the polygon. Also, the rectangle of minimum area enclosing the convex hull of an arbitrary plane curve is the same as the minimum area rectangle encasing the curve. Their method consists of two steps. Firstly, it determines the convex hull that encloses the given curve. Secondly, it rotates the convex hull such that one of its edges is collinear with a specific principal axis, say the X-axis, and then evaluates its bounding box area. This process is applied to all the edges of the curve. Finally, the rectangle of minimum area capable of containing this polygon is determined. Based on the idea of Freeman and Shapira[3], Martin and Stephenson[1] described a family of algorithms for solving problems such as whether a given object fits inside a rectangular box and the minimum bounding box for an object, both in two and three dimensions. Martin and Stephenson[1] proposed that the idea of Freeman and Shapira[3] could be extended to three-dimensional case. That is, the box of minimum volume enclosing a convex polyhedron has a face collinear with one of the faces of the polyhedron. Also, the box of minimum volume enclosing a convex hull of an arbitrary polyhedron is the same as the minimum volume box encasing the polyhedron. Thus, the object is oriented one at a time with each face of the polyhedron lying on say, the XY plane. The bounding box of the object and its volume are computed for each orientation. The smallest volume and its corresponding orientation are recorded.

In the above algorithm, the computational time is significant if the number of faces of the polyhedron is large because bounding box must be checked for each face. In the STL[11] computer model for rapid prototyping, the number of facets ranges from hundreds to millions. Furthermore, the real world is full of objects that are not polyhedron and convex. A conversion process[5] of these objects to convex polyhedra is needed. For a complex model, this conversion process is computationally expensive. Moreover, this method is approximate and the accuracy relies on the tolerance setting of the conversion process.

A new algorithm is proposed as an alternative to solve the problem with a simple approach. It simplifies the complicated 3D problem into a 2D problem. The algorithm is described in the following section.

THE ALGORITHM

The algorithm is used to determine the orientation of an arbitrary solid so that its bounding box volume is minimized. Relations between the 3D problem and the projected contours of the solid are introduced. An iterative method is constructed to deal with the projected contours.

Relations between the 3D bounding box and 2D projected contour

Problem: Given an arbitrary solid, how should the solid be positioned so that the volume of the bounding box is minimium? To obtain the minimum volume encasing box for a given arbitrary solid, we shall make use of the following definitions and theorems. Instead of showing the detail proofs of the theorems, an example is illustrated at the end of the theorems. The proofs are described in details in reference[12].

***DEFINITION*:** A bounding box is defined as a box whose body diagonal is delimited by the minimum X, Y, Z values and maximum X, Y, Z values of the given solid.

***AXIOM*:** Given an arbitrary solid model in 3D space, all orientations of the model can be achieved by rotating the model about any point in the 3D space.

***THEOREM 1*:** The areas of three mutually perpendicular faces of a given box are minimized if and only if the volume of the box is minimum. On the contrary, they are maximized if and only if the volume of the box is maximum.

The key issue is how to reorient a model to achieve Theorem 1. A new method is proposed (discussed later) which can be applied to an arbitrary model. During the rotation of a model about a point, its projected areas on the principal planes are inter-related. This means that the three bounding boxes are also inter-related and their relations are revealed in the following theorems.

***THEOREM 2*:** Given an arbitrary 2-D contour, let h and w be the height and width of its bounding box respectively. If the contour is rotated about the bounding box centre, the value of w will lag behind the value of h by $\pi/2$. At the same time, the values of h and w will repeat regularly by a period of π.

***THEOREM 3*:** Given the same arbitrary contour, if the contour is rotated about the bounding box centre, the bounding box area repeats itself by a period of $\pi/2$.

***THEOREM 4*:** Given the same arbitrary contour, if the height of the bounding box is equal to the width and the contour is rotated about the bounding box centre by an angle β, such that its bounding box area is minimum, then both the height and the width is minimum.

Let $A(\theta)$, $w(\theta)$ and $h(\theta)$ be the area, the width and the height functions at angle θ respectively. The theorem is stated as follows:

If $A(\beta)=\{A(\theta)\}_{\min}$ and $h(\beta)=w(\beta)$, then $h(\beta)=\{h(\theta)\}_{\min}$ and $w(\beta)=\{w(\theta)\}_{\min}$.

THEOREM 5: Following from the above theorem, if the contour is rotated about the bounding box centre by an angle β, such that its bounding box area is minimum, either the height or the width or both the height and the width of the bounding box is minimum. Let $A(\theta)$, $w(\theta)$ and $h(\theta)$ be the area, the width and the height functions at angle θ respectively. The reverse is also true. The theorem is stated as follows:

$A(\beta)=\{A(\theta)\}_{\min}$ if and only if $h(\beta)=\{h(\theta)\}_{\min}$ or $w(\beta)=\{w(\theta)\}_{\min}$ or both are true.

THEOREM 6: Given the same contour, the height of the bounding box of a contour will be equal to the width at a certain angle θ during rotation from 0 to $\pi/2$. The theorem is stated as follows:

$$h(\theta)=w(\theta) \text{ for an angle } \theta \text{ where } 0 \le \theta \le \pi/2$$

THEOREM 7: Given an arbitrary 2-D contour, let $h(\theta)$and $w(\theta)$ be the height and width functions of the bounding box respectively, where θ is the angle of rotation of the contour. $h(\theta)$ and $w(\theta)$ must be less than or equal to the length of the diagonal of the bounding box at any orientation. The theorem is stated as follows:

$h(\theta)\le\sqrt{h(\lambda)^2+w(\lambda)^2}$ and $w(\theta)\le\sqrt{h(\lambda)^2+w(\lambda)^2}$

where λ is any other angle of rotation of the contour.

THEOREM 8: Following form Theorem 7, if the area of the bounding box of the contour is minimum at angle β ($0<\beta<\pi/2$ by Theorem 3) and $h(\beta)$ is minimum (see Theorem 5), it remains minimum even though the contour is changed. This is true provided that $w'(\beta)=w(\beta)$ and $\{h'(\theta)\}_{\min}\le w(\beta)$ where $w'(\theta)$ and $h'(\theta)$ are the width and the height functions of the bounding box of the new contour respectively. The same applies if $w(\beta)$ is minimum instead of $h(\beta)$.

THEOREM 9: Given an arbitrary solid model, let R_x, R_y, R_z be the vectors parallel to the three principal axes X, Y, Z respectively. They also pass through the bounding box center of the arbitrary solid. Let P_{xy}, P_{yz}, P_{zx} be the three principal planes. If the model is first rotated about R_z until its projected bounding box area on the corresponding P_{xy} is minimum, one of the edges of the bounding box (i.e. height or width) will have achieved minimum. Without affecting the value of this edge, the model is further rotated about one of the two remaining vectors (i.e. R_x or R_y) until its projected bounding box area on the corresponding principal plane is minimum. One of the edges of the bounding box (i.e. height or width) will have achieved minimum. The model is further rotated about the vector, which is normal to the two minimum edges, until its projected bounding box area on the corresponding principal plane is minimum. The volume of the bounding box of the model so derived will thus be minimum. The same procedure can be applied if the model is rotated about the other two vectors (i.e. R_x or R_y) first.

Iteration process

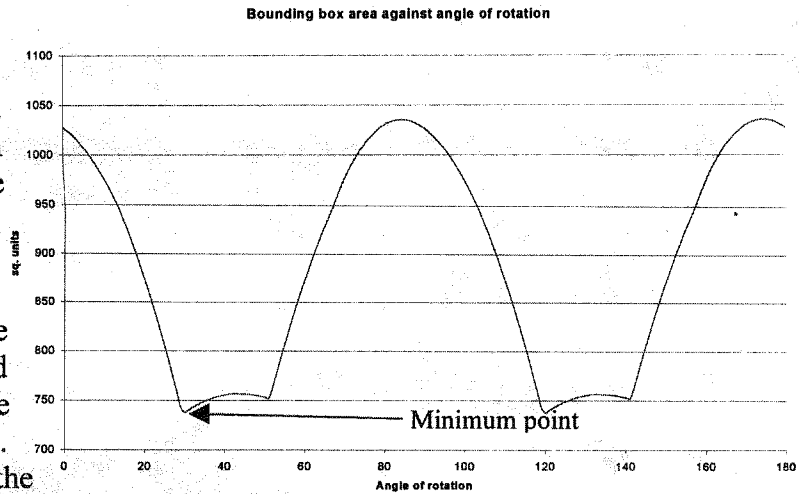


Figure 1. Projected bounding box area against angle of rotation.

A graph of the projected bounding box area of the model in Figure 3 on the XY plane against the angle of rotation is shown in Figure 1. By Theorem 3, the angle range being tested is π/2. To determine the absolute minimum located within this angle range, the following method is used. Within a small angle, the bounding box of the projected area of an arbitrary contour does not change much. Thus, the contour can be rotated by a small angle increment (δ), says 5 degree, within the angle range. This value of increment is sufficient for most contour patterns such that there is one minimum point located within the angle range only. Each time the bounding box of the contour and its area are computed, the smallest area and its corresponding angle (θ) are recorded. Thus, the absolute minimum projected area lies in the angle range (θ - δ) and (θ + δ). If β is the angle at which the projected bounding box area is minimum, we have

$$\theta - \delta < \beta < \theta + \delta .$$

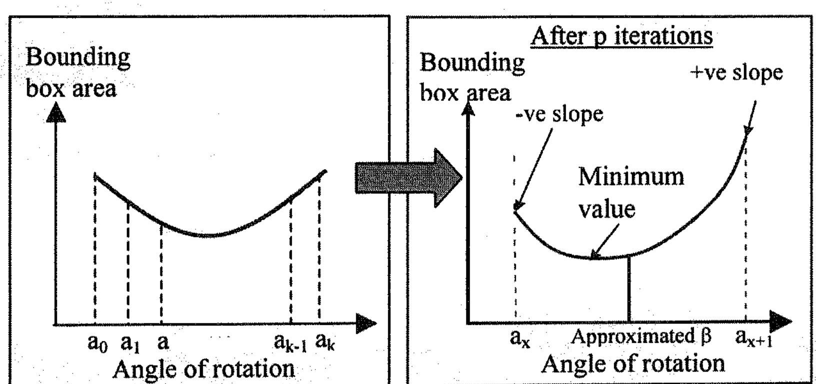


Figure 2. Bounding box area against angle of rotation.

Within the angle range, an iterative process can be started to determine β. The angle range can be divided into k equal segments at angles [a_0, a_1, a_2, a_3, ……, a_{k-1}] (see Figure 2). The projected contour is then rotated about its bounding box centre at angles [a_0, a_1, a_2, a_3, ……, a_{k-1}]. The corresponding slopes of the projected contour at these angles are recorded for further analysis. If there is a slope change across an angle pair (a_x, a_{x+1}) (see Figure 2), this pair will be used as the range for the next iteration until a predefined number of steps (p) is achieved. The angle (β) required for rotation is approximated by

$$\beta = a_x + 0.5 \times (a_{x+1} - a_x)$$

The accuracy of the iteration process is given by,

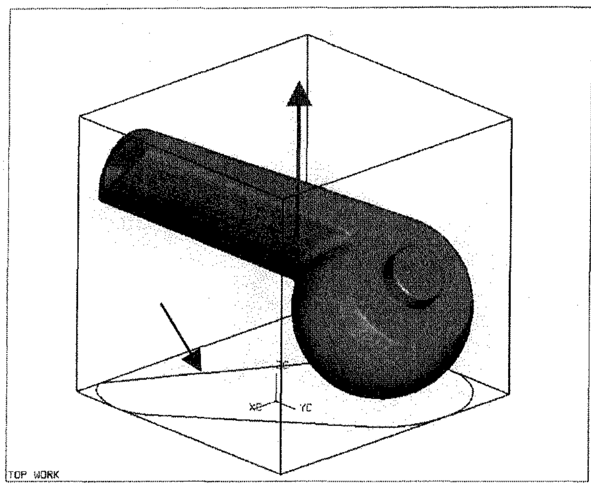
$$Accuracy = \frac{initial \ \ iteration \ \ range}{2 \times k^p} = \frac{2 \times \delta}{2 \times k^p} = \frac{\delta}{k^p}$$

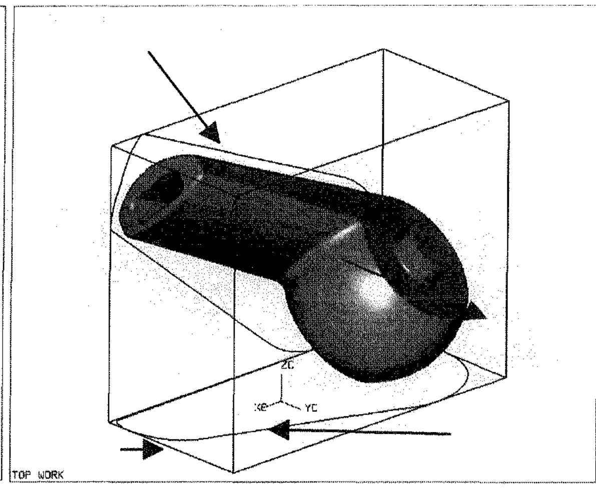
For example, if $\delta = 5$, $k = 4$ and $p = 5$,

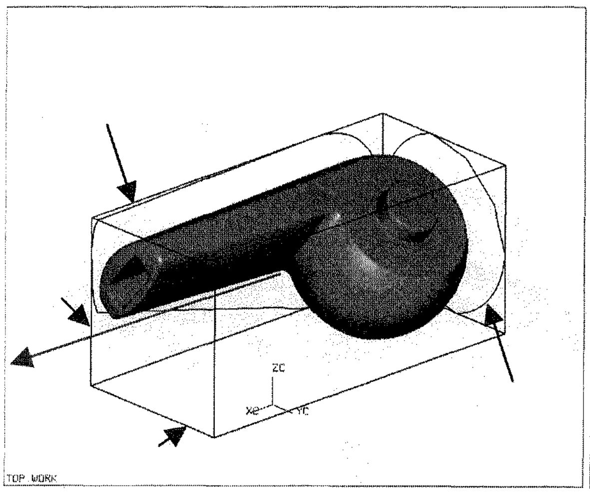
$$Accuracy = \frac{5}{4^5} \approx 0.005 \deg$$

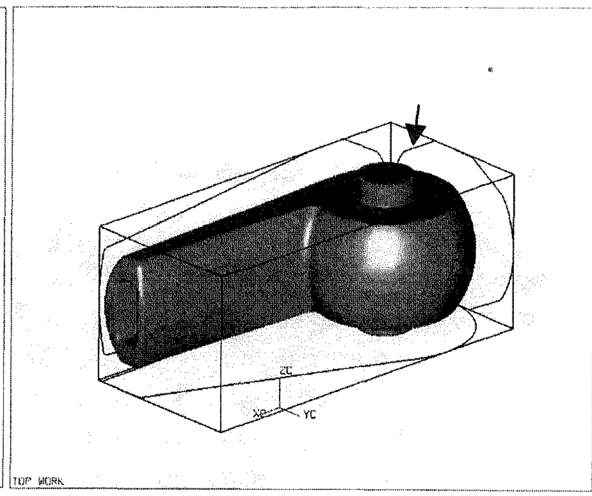
The iteration process is fairly accurate and efficient.

Examples









1. Projecting the model on the XY plane to find the 2D bounding box as shown in Figure 3.
2. Constructing an axis R_z normal to the XY plane and passing through the box centre.
3. Rotating the model about R_z until its projected bounding box area on XY plane achieves a minimum by the proposed algorithm (see Figure 3). The resultant model is shown in Figure 4 and we get the so called XY-bounding box with E_y is minimum in this case.
4. Now the bounding box edge that point to positive Y-axis direction is minimum. Therefore the axis R_y is used in next rotation. Repeat step 1, 2 and 3 for the ZX plane (see Figure 5). Hence we get the ZX-bounding box with E_z is minimum in this case.
5. Repeat step 1, 2, 3 for YZ planes which is formed by edge E_y and E_z. (see Figure 6)
6. By Theorem 9, the model is now oriented at a position which gives a minimum bounding box volume. One more example is shown in Figure 7.

	Model	Bounding box size (units)	Volume (cu. Units)	% reduction
Model in Figure 3	Original	38.6×31.6×30.7	31626	57.3
	Resultant	41.6×19.0×17.1	13520	
Model in Figure 7	Original	4.07×4.65×3.59	67.91	77.9
	Resultant	2.01×1.49×5.00	15.02	

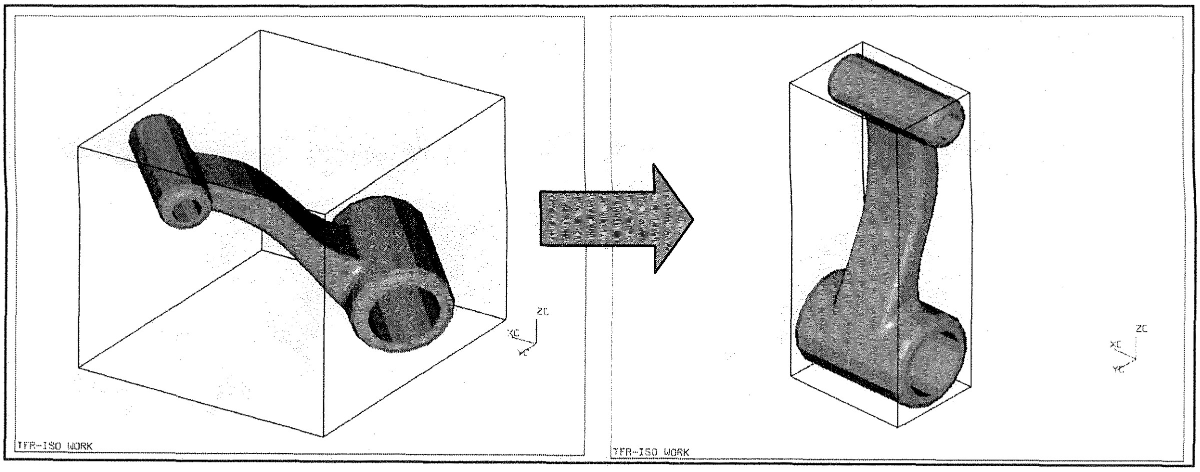


Figure 7. Example illustrating the proposed algorithm.

DISCUSSION

By the use of the above proposed algorithm, the minimum bounding box of an arbitrary model can be determined. The proposed algorithm is independent of model representation method, be it CSG model or B-rep. This is the main advantage over the one proposed by Martin and Stephenson[1].

The algorithm proposed by Martin and Stephenson[1] depends on the number of facets (f) generated from a model and also the number of edges (e) in each facet. The running time is linear $O(f \times e)$ without considering the pre-processing time, that is, the time requires to find the three-dimensional convex polyhedron. In reference (5), the authors showed that the running time for determining the three dimensional convex hull of a polyhedron is $O(n \times \log n)$ where n is the number of vertices. In the proposed algorithm, if a polyhedron is considered, the critical factor is the choice of angle increment (δ), number of iterations (p), and the number of subdivided element (k) in each iteration. The running time is $O[(k \times p)/\delta]$. It is very difficult to compare the functions of running time between the two processes because the parameters involved are different. However, the conversion process from an arbitrary model to a convex polyhedron is computationally expensive if the number of facets is large. On the contrary, the new algorithm can be applied directly to any model. The running time of the new algorithm is considered to be faster than that proposed by Martin and Stephenson[1].

In the case of complex shape solids for example, the one shown in Figure 8, it is rather difficult to determine its minimum bounding box. The data for the example built by SLS is shown in following table.

Model	Bounding box size (mm)	Build time (hrs)	% reduction
Original	65.03×62.89×65.97	4:19	23.9
Resultant	69.47×47.59×53.84	3:17	

Although the difference may not appear to be very significant, however, in certain application such as the rapid prototyping (RP) process, the time taken to slice the object positioned at the minimum box configuration would give rise to a significant saving in build time. This is particularly so for those RP processes whose build time is build height dependent, e.g. Selective Laser Sintering (SLS). The actual built part is shown in Figure 9.

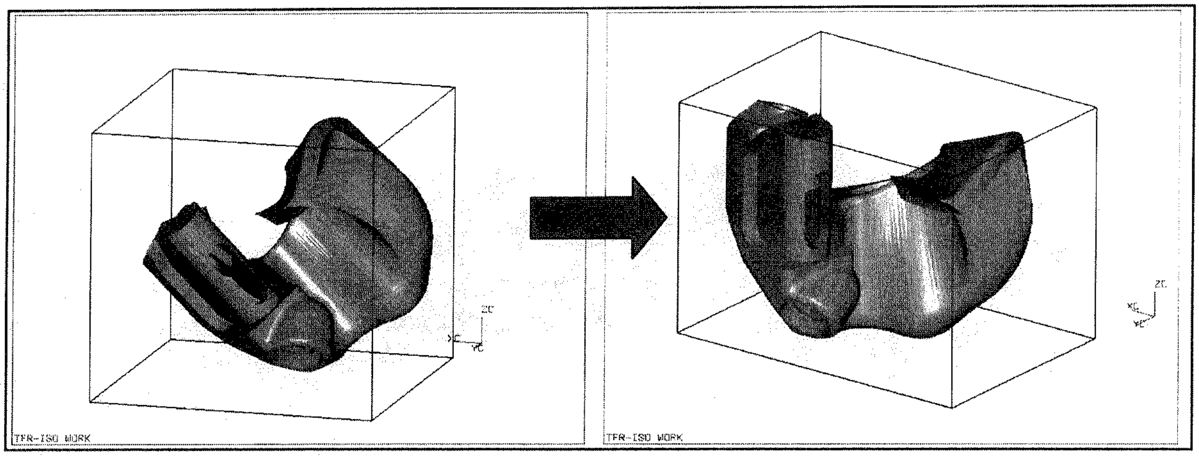


Figure 8. Complex solid from which minimum box can not be determined easily.

In the proposed algorithm, after obtaining the initial angle range using Theorem 3, the angle range for iteration is set by comparing the bounding box area in each rotation. Although this is valid for almost all model shapes, some special cases may be missed. Therefore, two other slightly modified approaches may be used.

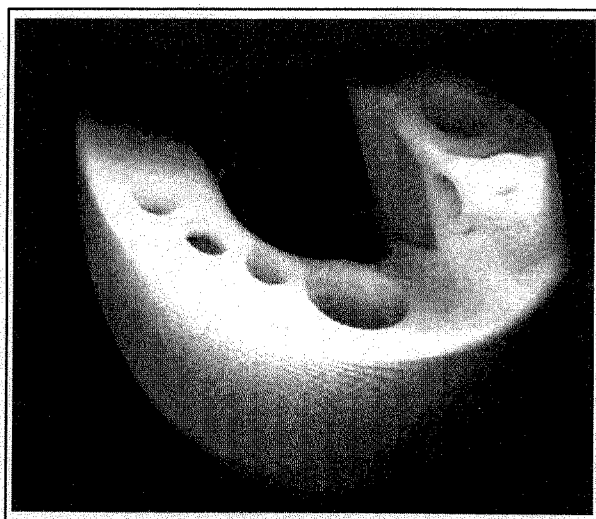


Figure 9. The actual built part by SLS.

Approach 1: The projected contour is rotated by a small angle increment within the initial angle range. After each rotation, its bounding box area and the slope at that point are evaluated. Since the sign of slopes around each turning point changes from positive to negative or vice versa, we can locate all minimum points by checking all the sign change of slopes. Through an iterative process, all minimum points can then be determined. Thus, the absolute minimum area and its corresponding angle can be found.

Approach 2: Instead of checking the sign change of slopes. A cubic spline curve can be fitted into those points and the absolute minimum can be determined from the curve equation.

The first approach is more general because it determines all the minimum points regardless of whether they are local or absolute. However, the running time depends on the number of minimum points. With more number of minimum points, the running time is increased but it does ensure that the derived area is an absolute minimum and its corresponding angle is the correct one. Comparatively, the second approach is faster and the running time depends on the angle increment. Although this method is the fastest, it compromises on the accuracy of the fitted minimum point. Some special contour pattern cases have also been noted. Firstly, if the projected contour is a regular polygon, the minimum bounding box areas of the contour during rotation will repeat regularly many times. Secondly, if the projected contour is a circle, the bounding box area of the contour is a constant throughout the rotation. Thirdly, if the projected contour is approximately round in shape, the bounding box area of the contour will vary within a very narrow range. All these case can be identified easily. This observation may be useful in further work for feature recognition.

CONCLUSION

In this paper, an alternative method for determining the minimum volume bounding box of an arbitrary solid has been suggested. By the use of the algorithm explained above, the minimum bounding box of any solid can be efficiently determined. The algorithm is also valid for a surface model as long as the bounding box of the projection of the arbitrary model can be determined[8]. The algorithm can be applied to determine the orientation of a model in packing problems, rapid prototyping and parting line determination[10]. Another interesting observation is that the reverse of the algorithm is also true. That is, the maximum bounding box of a model can be determined by reversing the algorithm.

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