Beam Structure Optimization for Additive Manufacturing based on Principal Stress Lines

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ABSTRACT

The benefits of component design with cellular structures have been demonstrated in a wide variety of applications. The recent advances in additive manufacturing and high performance computing have enabled us to design a product component with adaptive cellular structures to achieve significantly better performance. However, designing a product component with such structures, especially its shape and topology, poses significant challenges. Many approaches in topology optimization have been developed before for the purpose. In this paper, we present a novel structural optimization method based on the principal stress line analysis of a continuum domain. We first present the theoretical basis of our optimization method. We then discuss the properties of principal stress lines and their computation in a given design domain. Accordingly a novel structural optimization method is presented including size, shape and topology optimization. Related mathematical formulations and algorithms are also given for generating a beam structure with the minimum compliance. Three test cases are presented to illustrate the presented method.

KEYWORDS: Topology optimization, principal stress line, beam structures, minimum compliance.

1. INTRODUCTION

Over the last twenty years, *solid freeform fabrication* (SFF) has been developed based on the layer-based additive principle. Recent advances in material, process and machine development have enabled the SFF processes to evolve from prototype usage (rapid prototyping) to direct product manufacturing. Examples such as *Boeing*'s F/A-18 components and *Siemens*'s hearing aid shells have been reported for applications in aerospace and medical industries. SFF is a direct manufacturing process that can fabricate parts directly from CAD models without part-specific tooling or fixtures. Therefore, for the first time in history, we have a set of manufacturing processes that can cost-effectively fabricate truly complex 3-dimensional shapes. As identified in (Bourell, *et al.*, 2009; Hopkinson, *et al.*, 2006), a primary advantage of SFF is *its capability to allow revolutionary new designs when complex geometry is no longer a limiting factor*.

Even though SFF provides tremendous design freedom that was unavailable before, the capability of using SFF's unlimited geometric capabilities for better design is still limited and mainly untapped. As pointed out in (Bourell, *et al.*, 2009), it is crucial to develop new design methods and related CAD tools in order to fully utilize the design freedom provided by SFF. Since beam structure is a type of design that can well demonstrate the SFF's geometric capability, we focus on the design optimization of beam structures in this paper, including those of size, shape/geometry and topology.

Topology optimization of trusses is a classical subject in structural design. The study of fundamental properties of optimal grid-like continua was made by Michell (1904). In the last three decades, a considerable amount of work has been carried out on structural optimization. Two developed computational methods that are the most well-known for truss topology design are the Homogenization and the Ground Structure methods. More detailed description of them and related methods are given in Section 2. Even though the two methods are well-established, problems such as the generated design needs to be interpreted and in many cases unnatural to users have also been noticed. In addition, both methods are computationally expensive; hence it is rather difficult for designers to control the generated results. Motivated by Michell's work, we present an intuitive method of designing beam structures with a minimum compliance or maximum stiffness. Our method is based on the principal stress line analysis of the given design domain. The mathematical foundation of our method is presented in Section 3. A high-level comparison of our approach with the homogenization and ground structure methods is given in Figure 1. We believe our approach is much simpler and faster. In addition, the generated structures have no issues that are common to the homogenization and ground structure methods such as gray area and disconnected structures as pointed out by (Lu and Kota, 2006).

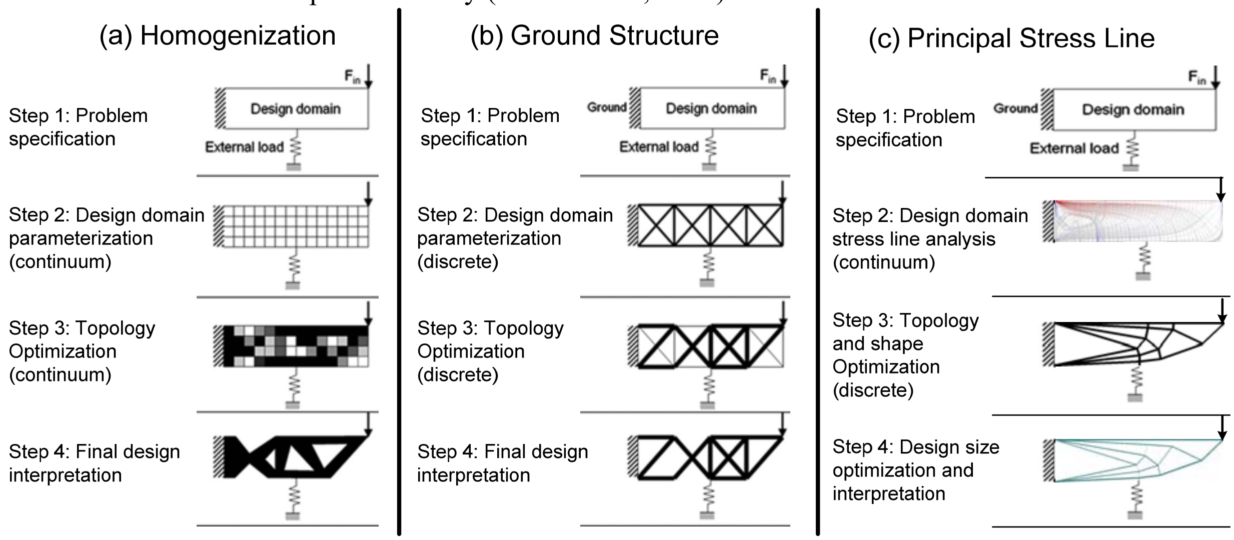


Figure 1. A comparison of our method with Homogenization and Ground Structure methods (Lu and Kota, 2006).

2. RELATED WORK

Topology optimization is a classical subject in structural design. Introductions to truss topology problems can be found in (Topping, 1993; Kirsch, 1993; Rozvany, *et al.*, 1995a; Achtziger, 1997; Bendsøe and Sigmund, 2002). Topology optimization of discretized and continuum structures are two broad categories in the structural optimization. We focus on discrete structure optimization in this paper. Hemp and Chan (1970), and Dorn, Gomory and Greenberg (1964) considered a ground structure to overcome the infeasibilities of Michell structures. Given the same design domain, the analogous boundary conditions and external loads, they obtained the trusses coincide with the principal stresses directions of an optimal continuum structure (Achtziger, 1997). Nowadays the ground structure method is a well-known approach in the discrete topology optimization.

Ground structure is composed of uniform spaced nodes connected with each other by boundary conditions and external loads or forces. The ground truss structure is thought to encompass the potential optimal structure. Introductions to ground structure approach can be found in (Topping, 1993; Achtziger, 1997; Bendsøe and Sigmund, 2002). The numerical computational theories on ground structure approach are mainly founded on minimization of compliance or maximization of stiffness. This objective function has been utilized in many literatures (Bendsøe and Sigmund, 2002; Achtziger, 1997; Bendsøe, 1995; Rozvany, *et al.*, 1995a; Svanberg, 1990; Svanberg, 1994).

In order to solve this objective function of minimization of compliance, linear or non linear programming techniques have been developed (Achtziger, 1997; Achtziger, *et al.*, 2008; Achtziger and Stolpe, 2009). There are some other numerical computational approaches used to find the optimal truss structure from a ground truss structure (Hajela, *et al.*, 1993; Xie and Steven, 1997). Node positions in a ground structure are to be optimized as well as topology and truss bar cross sectional size optimization. This further node positions optimization is called geometric approach (Topping, 1993). In geometric approach, the node coordinates are also considered to be variables as well as bar cross sectional size. Research on both topology and geometry of ground structures can be found in (Ben-Tal, *et al.*, 1993;

Achtziger, 2007). The complexity of ground structure approach is $O(n^2)$, where n is the number of the nodes. When n is large, the ground structure is very dense; and the numerical computation of LP or nonlinear LP techniques is unstable; and some unreasonable structures are obtained (Burns, 2002).

Rozvany, *et al.* (1995b) noticed that many literatures focused on truss structures rather than beam structures. They also pointed out that the beam structures are more practical than truss structures. For truss structures, pin-jointed by only considering the axial forces and buckling is unavoidable. Beam structures are rigid-jointed and considering the shear forces which are matched with the engineering purpose. Theories on beam structure optimization can be found in (Rozvany, 1994; Rozvany and Prager, 1976). However, these theories are targeted for the continuum or partial continuum beam structure optimization. They cannot be simply applied in numerical computation. Some people combined the ground structure approach with the continuum based material optimization method to design the beam structures with joints (Fredricson, *et al.*, 2003; Fredricson, *et al.*, 2003; Fredricson, 2005; Kim, *et al.*, 2008). These beam structures with joints are optimized with their sizes and can be manufactured directly. They are called compliant assemblies. Basically their physical model is a 3D solid one. Either sensitivity analysis or SIMP (simple isotropic material with penalization (Bendsøe and Sigmund, 1999) are utilized to optimize the size of each beam. Their optimization methods are continuum based.

Motivated by simplifying the complexity of ground structure, we proposed a principal stress line method for the beam structure design. For the purpose, we also extended uniform strain energy density from truss structure to beam structure topology optimization. The remainder of the paper is organized as follows. Section 3 introduces the topology optimization problem and related mathematical formulations. The principal stress lines and their properties are also discussed in the section. Section 4 presents an overview of the principal stress line method. Details of the method are then presented in the following sections. Section 5 discusses the numerical method to compute the principal stress lines of a given design domain. Section 6 discusses the initial structure generation for a given design domain. Section 7 presents our size optimization method based on the uniform strain energy density principle. Section 8 discusses the topology growth of a candidate beam structure based on the principal stress lines. A test example is presented in Section 9. Finally Section 10 concludes the paper.

3. PROBLEM FORMULATIONS AND OUR APPROACH

3.1. Problem Formulation of Beam Optimization

The optimization problem considered in the paper is *to minimize the compliance on the total structural volume of a beam structure under static loads and constraints with the cross-sectional areas, the nodal coordinates, and the beam connections as design variables*. Hence all the three types of optimization problems are considered including (1) size optimization by changing cross-sectional areas; (2) shape optimization by changing the nodal coordinates; and (3) topology optimization by changing the beam connections. Even though most discrete optimization literatures consider truss structures, we focus on beam structures since they are more appropriate for SFF.

Based on the linear elasticity theory (Eschenauer and Olhoff, 2001), the beam structure in our problem follows the basic equations: $\prod = U - W = \frac{1}{2}\mathbf{u}^{\mathbf{T}}\mathbf{K}\mathbf{u} - \mathbf{F}^{\mathbf{T}}\mathbf{u}$, (3.1)

where **u**, **K** an **F** are the global displacement vector, the stiffness matrix and the load vector. Hence, the problem of minimizing the work done by external forces at equilibrium can be formulated as:

$$\min_{\chi \in B} W(\mathbf{u}) \text{ subject to } \mathbf{Ku} = \mathbf{F}, \quad (3.2)$$

Where B is a specific volume of body given by $B = \left\{ \chi \left| \int_{\Omega} \chi dx = V_{fixed} \right. \right\}$, and χ denotes the design variable. By applying the equation $U = \frac{1}{2}W$ at equilibrium, the given optimum problem can also be alternatively written as:

$$\min_{\chi \in B} U(\mathbf{u}) \text{ subject to } \mathbf{Ku} = \mathbf{F}, \quad (3.3)$$

3.2. Michell Problem and Principal Stress Lines

Michell (1904) first studies the fundamental properties of optimal grid-like continua and contribute exact analytical solutions for several well-known truss structures. One of them is Michell-cantilever, which is shown in Figure 2. Modern layout theory was founded by Prager and Rozvany (Prager and Rozvany 1977). Rozvany, *et al.* (1995a) also state that Michell trusses have the maximum stiffness for a given volume. According to Strang and Kohn (1983), the Michell problem is defined as follows:

$$div\sigma = 0 \text{ in } \Omega, \ \sigma : n = f \text{ on } \Gamma \tag{3.4}$$

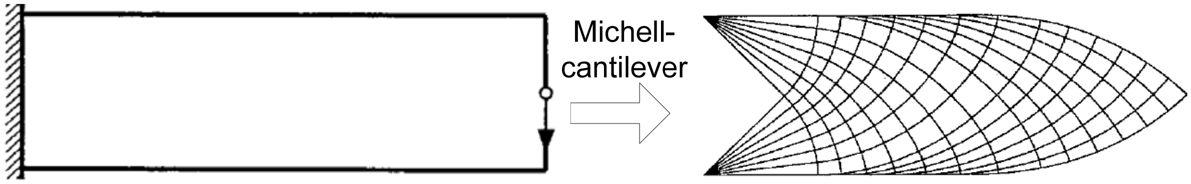


Figure 2. Least-weight truss for a cntilever with a point load (Rozvany 1998).

Hence, one can imagine that the bars of a truss-like continuum are placed in the directions of principal stresses. That is, the Michell structure composes of infinite number of truss bars perpendicular to each other at the intersection (Strang and Kohn, 1983; Prager, 1974). And the slope of the principal stresses with respect to the x-axis is given in (Hill, 1950) as:

$$\tan 2\theta = \frac{2\tau_{xy}}{\sigma_x - \sigma_y} \tag{3.5}$$

From (Pilkey, 2002), we have the stress strain relation as:

$$\begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix} = \frac{E}{(1+v)(1-2v)} \begin{bmatrix} 1-v & v & 0 \\ v & 1-v & 0 \\ 0 & 0 & \frac{1-2v}{2} \end{bmatrix} \begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{bmatrix} \tag{3.6}$$

where E is the elastic or Young's modulus; v is Poisson's Ratio; $\varepsilon_x, \varepsilon_y$ are strain in x and y direction respectively; γ_{xy} is shear stress in the xy plane. Hence we have

$$\tan 2\theta = \frac{\gamma_{xy}}{\varepsilon_x - \varepsilon_y} \tag{3.7}$$

Therefore the principal stress lines are defined as the two orthogonal families of curves whose directions at every point coincide with those of the minimum shear stress in a state of plane strain.

3.3. Principal Stress Line Analysis for Structural Optimization

From the conditions of elastic equilibrium, given the design domain, external forces and boundary conditions, in a finite element system of isotropic material, the following properties hold:

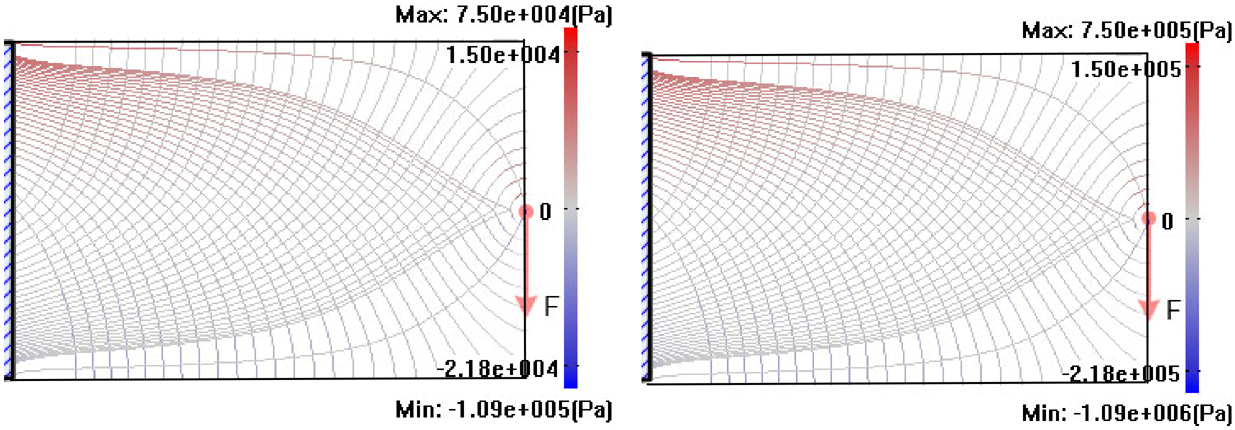
Property 3.1 The displacement vector $\mathbf{u}$ is proportional to the external force vector $\mathbf{F}$.

Property 3.2 The displacement vector $\mathbf{u}$ is inversely proportional to the stiffness matrix $\mathbf{K}$ or the elastic modulus **E**.

Property 3.3 The direction of the principal stress is not related to the scaling of the external forces, nor the material type for an isotropic material within the range of elastic deformation.

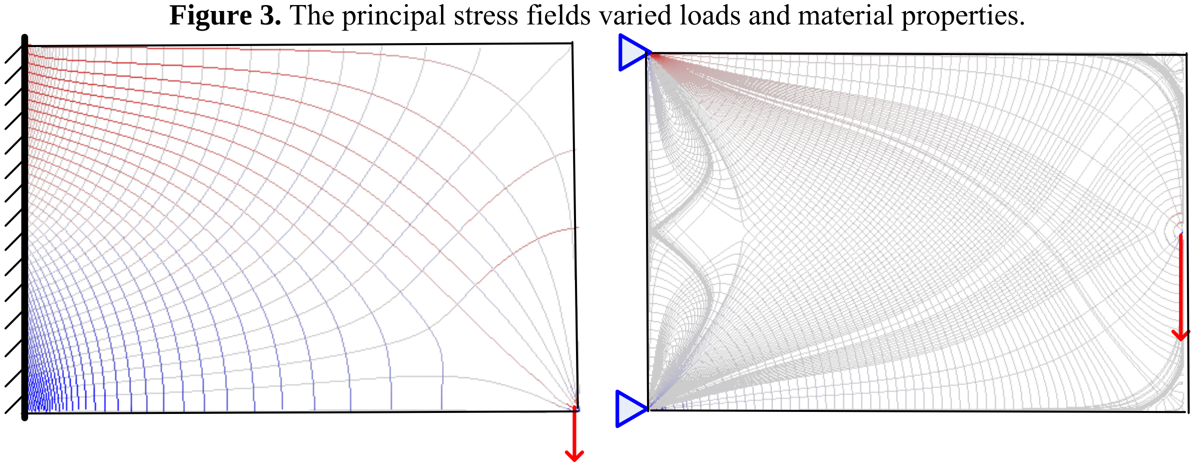
Property 3.4 The principal stress field is mainly related to the topologic variables of the given structural design such as the position of external forces and the types of constraints.

As an example, we present the principal stress field of a simple cantilever beam as shown in Figure 3. For a single load with various sizes and different material properties, the principal stress lines are virtually the same. This suggests that their optimal topology and shape design should be identical. Hence both load sizes and material properties will not affect topology design results. We also present the principal stress field for the same cantilever beam with different load or constraint positions. As shown in Figure 4.(a), the external force **F** is moved from the middle to the bottom of the right edge. As shown in Figure 4.(b), two point constraints are used instead of a line constraints in Figure 3. Since the principal stress fields have been changed, their optimal topology and shape design should also be different.



(a) Steel, E=2.0e11[Pa], F=10[N] (b) Aluminium, E=7.0e10[Pa], F=100[N]

Figure 3. The principal stress fields varied loads and material properties.



(a) Different load position (b) Different types of constraints

Figure 4. Principal stress lines distribution in the continuum design domain.

4. THE PRINCIPAL STRESS LINE METHOD

Motivated by Michell structures and the properties of principal stress lines, we developed a principal stress line method for designing a beam structure with the minimum compliance and maximum stiffness under any given loads and constraints. The optimization of size, topology, and shape of beam structure are all considered in our method. The design process of our method is shown in Figure 5 by using a single loaded Michell structure as illustration. There are five major steps including: (1) design domain specification, (2) principal stress lines computation, (3) initial structure generation, (4) size optimization, and (5) topology growth.

For a given design domain specified by a user, the finite element method (FEM) is used to analyze the design domain based on the specified loads and constraints, the material information, and the maximum space that the resulting structure can occupy. Accordingly the principal stress lines can be computed and visualized. An initial structure is generated to connect all the constraints and loads. Based on the initial structure, size optimization is then carried out, which modifies the cross section area of each beam for improved performance. More principal stress lines can be added in topology growth, which will accordingly add more nodes and beams to the beam structure. Hence a new beam structure with different topology and shape will be generated for size optimization. The process is repeated until no performance improvement is found or a limit on number of beams has been reached. Sections 5~8 will explain in detail each of the five steps in the above process.

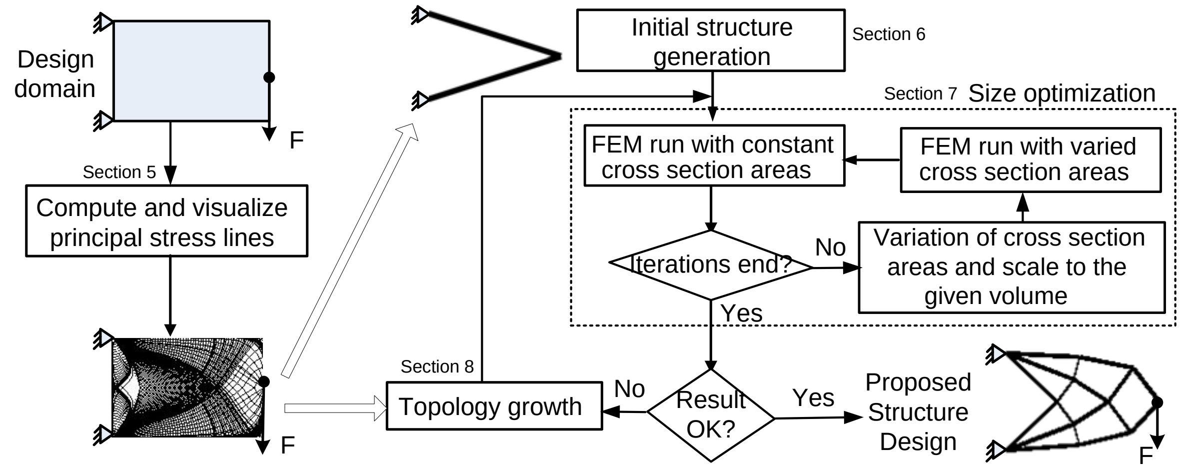


Figure 5. An overview of the principal stress line method for beam structure design.

5. PRINCIPAL STRESS LINE COMPUTATION

Hegemier and Prager (Hegemier and Prager, 1969) utilizes the mathematical equations to compute the Michell truss bars based on the analytical computation of the directions of principal strains for a single loaded truss structure. The similar complementary slip lines are computed analytically in (Hill, 1950). In this paper, we develop a general numerical computation method to compute the directions of the principal stress lines. In addition to simple cases with a single load such as Michell truss bars, our numerical computation method can handle general cases with more complex boundary conditions and various external loads.

5.1. Computing Principal Stresses at Any Point

The given design domain with loads and constraints is first analyzed based on the finite element method. As shown in (3.6) and (3.7), the sizes and directions of the principal stresses are well defined for any given point in the stress field of a continuum 2D solid. However, in the 2D plane static analysis results generated by a FEM system (we used COMSOL3.2: www.comsol.com), the finite elements are a set of triangles and all the post-processing data of normal stresses σ_x, σ_y and shear stress τ_{xy} correspond to triangle vertices. In order to query the post-processing data in any position in the design space, we compute the stresses based on linear interpolation. That is, suppose the stresses at position P is needed and assume P is located in a triangle element $\Delta P_1 P_2 P_3$. The corresponding post-processing data (i.e. σ_x, σ_y or τ_{xy}) for P_1, P_2 and P_3 are D_1, D_2 and D_3 separately. The post-processing data D at the position P can be computed as: $D = uD_1 + vD_2 + wD_3$ (5.1)

where $u = \frac{\left\| \vec{PP_2} \times \vec{PP_3} \right\|}{\left\| \vec{P_1P_2} \times \vec{P_1P_3} \right\|}, v = \frac{\left\| \vec{PP_3} \times \vec{PP_1} \right\|}{\left\| \vec{P_1P_2} \times \vec{P_1P_3} \right\|}, w = \frac{\left\| \vec{PP_1} \times \vec{PP_2} \right\|}{\left\| \vec{P_1P_2} \times \vec{P_1P_3} \right\|}$. Hence the principal stresses σ_1, σ_2 and the corresponding direction parameterized by θ at P can be computed.

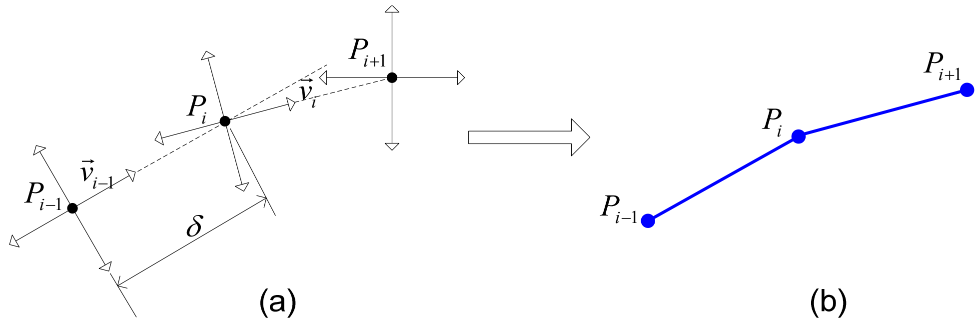


Figure 6. Principal stress line search based on stress field points.

5.2. Generating 2D Principal Stress Lines in a Continuum Domain

Based on the principal stresses computed for a given point in the design domain, we will have four principal directions (i.e., in θ, $90^\circ + \theta$, $180^\circ + \theta$, and $270^\circ + \theta$). Hence four principal stress lines can be generated beginning from a stress field point with the four principal directions. For simplicity and without loss of generality, we will discuss the principal stress line search strategy in one direction. As shown in Figure 6.(a), from point P_{i-1}, we can search one of its principal stress direction $\vec{v}_{i-1}$ and compute point P_i for a given approximation step δ. Accordingly the resulted principal stress line is shown in Figure 6.(b).

Hence, we have the following iterative equations:

$$\begin{cases} P_0, \vec{v}_0 & i = 0 \\ P_i = P_{i-1} + \delta \cdot \vec{v}_{i-1} & i \geq 1 \end{cases} \tag{5.2}$$

Where $P_0, \vec{v}_0$ are the starting position and the principal direction; δ is the search step; $\vec{v}_i$ is one of the four principal directions at position P_i closest to the search direction $\vec{v}_{i-1}$. The closest direction selection is finished by comparing the angle with the search direction $\vec{v}_{i-1}$.

Sufficient number of starting points is needed for generating principal stress lines that are dense enough to cover the entire design domain. We used two strategies in generating principal stress lines that are uniformly distributed in the design space.

(1) For a simple design space, we add a guide curve and sample the curve uniformly. The sampling points are used as the starting points to generate the principal stress lines.

(2) For a complex design space, we used the design space boundary as the guide curve and sample the curve uniformly. The sampling points are used as the starting points to generate the principal stress lines.

6. DOMAIN SPECIFICATION AND INITIAL STRUCTURE GENERATION

As shown in Figure 5, our method starts from the user specifying the loads and constraints, and also the maximum domain space that the resulting structure can occupy. Based on the computed principal stress lines, the user can identify the principal stress lines that connect the constraints and loads. We called such lines *skeleton principal stress lines*. An initial structure can then be generated from the *skeleton principal stress lines*. The generation of the initial structure can be performed automatically or by the user. For example, Figure 7.left shows the design domain for the Michell cantilever. The physics properties we used in the test are: H = 1.0 m; F = 1000 N; Young's modulus E = 2.0e11 Pa; Poisson's ratio ν = 0.33; Density = 7850 kg/m^3; Shape of a beam's cross section = square; Given volume of the material = 0.015 m^3. Figure 7.right shows the computed principal stress lines. The *skeleton principal stress lines* are also shown that connect the load and constraints. Accordingly an initial structure can be generated.

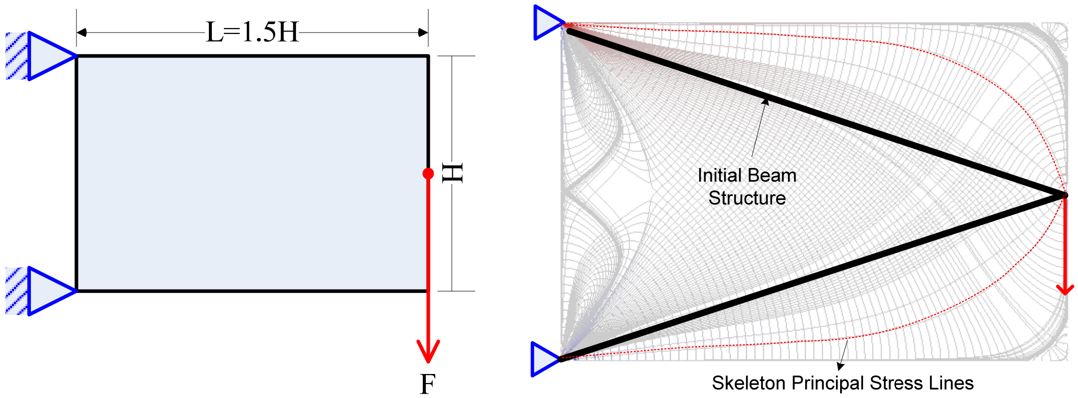


Figure 7. The design domain and initial structure of a Michell cantilever beam case.

Notice after a candidate principal stress line is selected, we use a straight line connecting its two nodes in the beam structure instead of approximating the exact stress line by a set of polylines. This is because, for two given nodes in a beam structure, a straight line connecting them will lead to a smaller compliance of the structure than any other curves to connect them.

7. SIZE OPTIMIZATION OF BEAM STRUCTURES

In our beam structure design, we follow energy principles which present the basis for both topology and size optimization of discrete and continuum structures.

7.1. Theoretical Basis – Axiom of Uniform Strain Energy Density

Based on the assumption of the positive definiteness of the global stiffness matrix, Achtziger (Achtziger, 1997) proved that the strain energy density is uniform among the truss structure. However, no references were found on beam structures. Here we will extend some conclusions of topology optimization of truss structures to the topology optimization of beam structures.

As shown in (Achtziger, 1997), for the maximum stiffness topology optimization of loaded truss structure, the optimization problem (3.2) can be reformulated as:

$$(P_{strEn}) \qquad \max_{u \in R^n} \min_{1 \le i \le m} \{ \mathbf{f}^T \mathbf{u} - \frac{V}{2} \mathbf{u}^{\mathrm{T}} \mathbf{K}_{\mathbf{i}} \mathbf{u} \} \tag{7.1}$$

Where $\mathbf{f}$ is a vector consisting of the external loads; $\mathbf{u}$ is the displacement vector consisting of all displacement components for all bars i and $1 \le i \le m$; $\mathbf{K}_{\mathbf{i}}$ is the stiffness matrix of the ith bar. The proof can be extended to the minimization compliance or maximization stiffness topology optimization of discrete beam structures. To satisfy the optimality conditions in the uniform strain energy density, we know:

$$x_i^{new} := x_i \frac{\frac{1}{2} u(x)^T K_i u(x)}{\max_{1 \le j \le m} \left\{ \frac{1}{2} u(x)^T K_j u(x) \right\}} \tag{7.2}$$

For all $i = 1, \ldots, m$ beams, $\frac{1}{2} u(x)^T K_i u(x)$ is the strain energy density. When a beam structure achieves the equilibrium, the strain energy density is uniform.

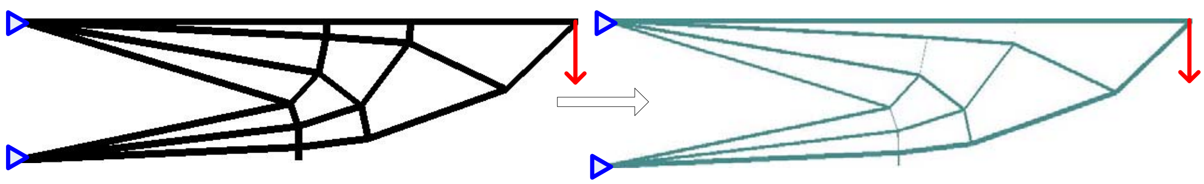
7.2. Computation Approach

We used 3D Euler Beam module in *COMSOL Multiphysics* 3.2 as the FEM tool to compute and analyze the beam structures. In order to exploit the post-processing data from COMSOL and do the further computation and optimization work, we developed a set of automation tools for the analysis process based on a programming language, COMSOL script, provided by COMSOL. We also use the COMSOL script language to compute the strain energy of a beam structure. After a beam structure is analyzed, the strain energy is computed and saved in post-processing data. The strain energy density is expressed as strain energy per unit length of a beam in COMSOL. As shown in Section 7.1, the objective in size optimization is to achieve the maximal stiffness of the beam structure. Based on the method of uniform strain energy density, to maximize stiffness also means to minimize the strain energy of the beam structures. In addition, the beam structure achieves the minimal potential energy when the strain energy density is uniform in the entire beam structure.

We define the strain energy density as strain energy per unit volume of a beam, denoted by κ with unit of J/m^3. Hence we can optimize the cross section size of each beam for a given beam structure based on a process that is similar to the *Fully Stressed Design Stress-Ratio* Method in truss structures (Topping, 1993). However, different criteria are used. That is, instead of stress ratio, we use the strain energy density ratio to modify the cross section area of beam members after each stress/strain analysis of beam structures. In each iteration, we use:

$$A_{inew} = A_{iold} \cdot \left(\frac{\kappa_i}{\max\limits_{1 \le i \le m} \{\kappa_i\}} \right)^{\frac{1}{2}} \quad (7.3)$$

After cross section areas of all beams are resized by the strain energy density ratio method, all cross section areas of beams are scaled to fit the given volume X_V (note zero cross section beams still have the threshold cross section area ratio). With the resized beam structure, a new FEM stress/strain calculation is carried out, in which the cross section area in the bigger strain energy density beams gets bigger and bigger(in a relative way), and the cross section area in the smaller strain energy density beams gets smaller and smaller. The process repeats until the strain energy distribution in the whole beam structure is close to uniform. An example of size optimization based on the above process is shown in Figure 8.



(a) A beam structure with uniform cross section; (b) resulted beam structure after size optimization

Figure 8: An test example of size optimization based on uniform strain energy density.

8. TOPOLOGY GROWTH BASED ON PRINCIPAL STRESS LINES

The theoretical Michell structure has infinite number of bars and each bar is infinitely small. Hence such exact analytical solution is not practical for a beam structure design. Instead we would like to balance the number of beams and the performance of beam structure. For the purpose, we present a topology growth method based on the principal stress lines. From an initial structure, the main process of our method is:

(1) Identify new principal stress lines that can reduce the approximation errors between the beam structure and the principal stress lines the most;
(2) Use the identified principal stress lines to compute a set of intersection points as the positions of inserted nodes (shape optimization);
(3) Construct a beam structure by connecting the computed nodes following the connection of the principal stress lines (topology optimization).
(4) Optimize the cross section size of each beam for the constructed beam structure (size optimization). The strain energy of the whole beam structure can be computed, which can measure the stiffness of the beam structure.

The above process repeats until desired performance is reached or the maximum number of beams is exceeded. This is illustrated in Figure 9 based on a Michell cantilever beam as shown in Figure 7. As shown in Figure 9, based on the initial structure in iteration 0, the maximum errors from the principal stress lines and the related beams are P_{1a} and P_{1b}. Accordingly two new principal stress lines are added which will lead to a new node P_{1c}. Therefore, a new beam structure can be constructed in iteration 1.

After size optimization, the strain energy of the new beam structure is 3.61 mJ, which is significantly reduced from that of the beam structure in the previous iteration (4.15 mJ). Hence the stiffness of the beam structure has been increased for the same amount of material. The process can be repeated with more nodes and beams added. In the above topology growth process, it can be noticed that as more principal stress lines are used, the beam structure is closer to the Michell structure. The strain energy of the entire structure is getting smaller. However, the performance improvements are getting smaller after several iterations.

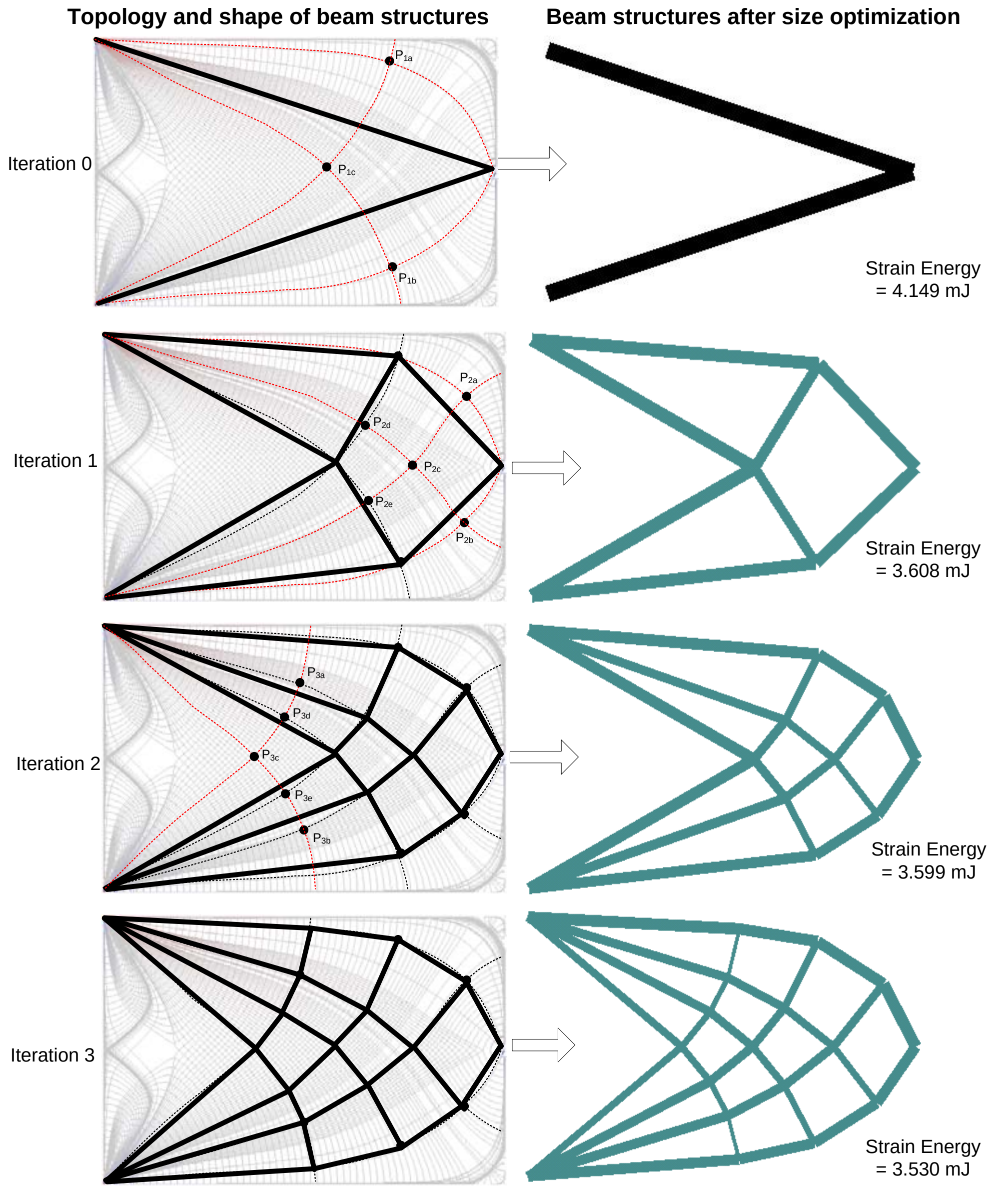


Figure 9. The topology growth of a Michell cantilever beam structure in 4 iterations.

In our beam structure design process, we use the principal stress line as a bridge between the continuum design domain and the discrete beam structure design. This is different from other methods such as Homogenization, Ground Structure, and other expansion and growth methods for truss structure optimization (Kirsch, 1997; Martinez 2009). Compared to them, our method is dramatically faster. In addition, our method is easier to control. Hence the user can easily terminate or add more beams in the designed beam structure. However, a limitation of our method is that it can only be used for the minimum compliance or maximum stiffness structural optimization problem.

9. TEST EXAMPLE: BRIDGE STRUCTURE

To demonstrate the presented principal stress line method, another example of a bridge structure is presented. Similar to the Michell cantilever beam, the bridge structure is also a classical topology optimization test case. The design domain of a bridge structure is shown in Figure 10.left. The domain is fixed at the two end points of the bottom and a concentrated force is added in the middle of the bottom edge. The principal stress lines of the design domain based on the method discussed in section 5 are shown in Figure 10.right.

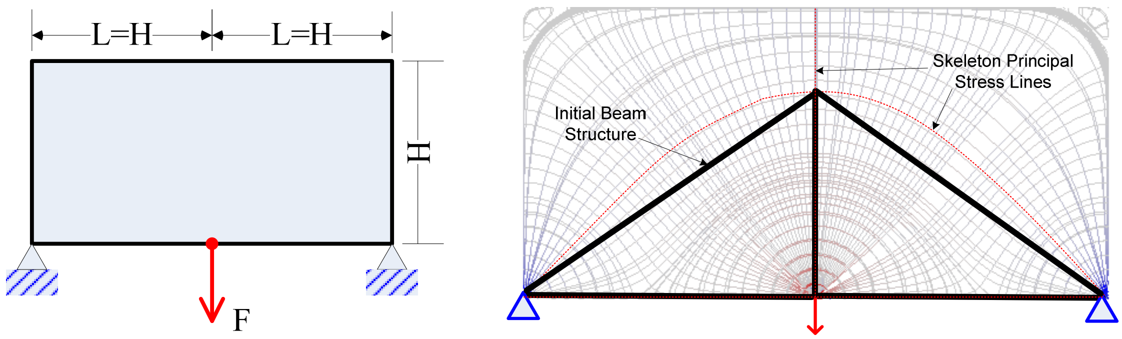


Figure 10. a bridge like design domain and the principal stress lines.

Based on the computed principal stress lines, the *skeleton principal stress lines* to connect the constraints and loads are shown in Figure 10.right. Notice the principal stress lines at two constraints have 45 degrees tangent with the horizontal axis. Accordingly, an initial structure can be generated which is also shown in the figure. The topology growth process of the structure is shown in Figure 11. After three iterations, the strain energy of the beam structure decreases from 1.33 mJ to 1.11 mJ. Hence the stiffness of the beam structure has been increased for the same amount of material used in the structure.

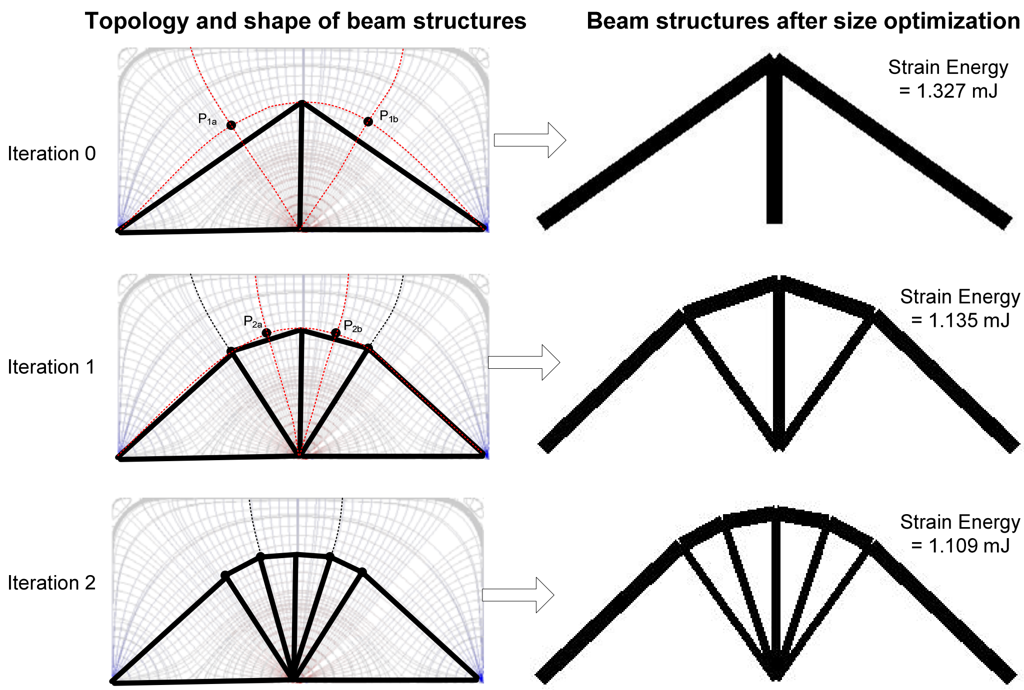


Figure 11. The topology growth of a bridge structure in 3 iterations.

10. CONCLUSIONS

For the minimum compliance design of beam structures, we presented a new structural optimization method based on principal stress lines. The mathematical basis of our discrete beam structure optimization method has been presented. We also presented a numerical method for computing the principal stress field of any given design domain. Accordingly a topology growth method for refining beam structures has been presented. In addition, based on the uniform strain energy density proposition in discrete truss structure, we extended the axiom of strain energy density to the discrete beam structure. The axiom of strain energy density is used in our size optimization of beam structures. Three examples were given to illustrate the design process of the principal stress line method. The results have demonstrated the effectiveness of our method. Compared to the ground structure method, the principal stress line method is much faster and easier to control. It is also more predicable and can overcome the numerical computation instability and results of unreasonable structures when the ground structure is highly dense.

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