# Contour Following for Scanning Control in SFF Application : Control Trajectory Planning

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## Abstract

Geometric contour following for scanning control in SFF application is used to refine the boundary of the parts for increasing the accuracy or to develop the capability to arrange various scanning directions and paths for improving the part strength. The scanners must be driven to follow the prescribed path as fast as possible, limited by available torques. In this paper the minimum time optimal control problem with specified path and limited control torque is formulated. According to the trade-off between various requirements, a control strategy is studied.

## Introduction

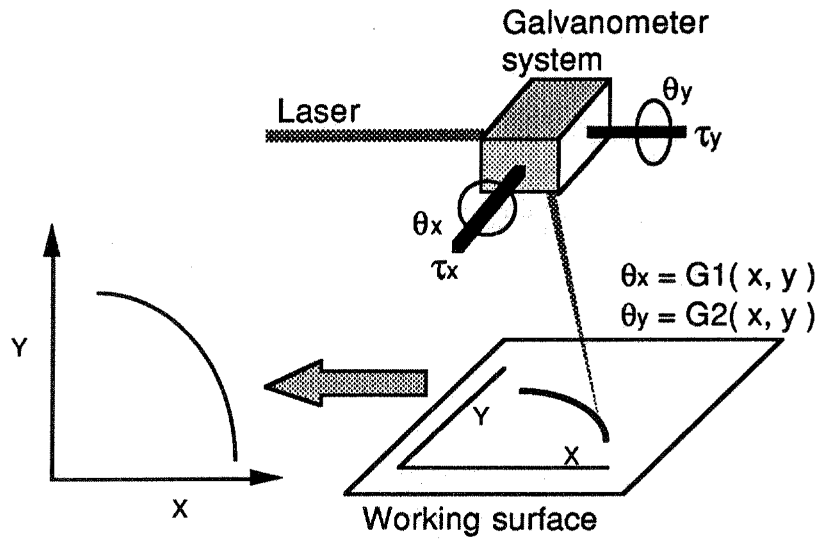
Geometric contour following for scanning control in SFF is used to refine the boundary of the parts for increasing the accuracy or to develop the capability to arrange various scanning directions and paths for improving the part strength. The raster mode and vector mode in scanner application are common in industry. However, the former is not appropriate to this application, the latter is too slow. Thus, it is important to develop a control strategy such that the scanners are driven to follow the prescribed path as fast as possible without exceeding the available torques. In this paper, the minimum time optimal control with specified path and limited control torque is formulated. One special point in SFF application is that the control torques and specified path are in different spaces shown in fig#1, like a two-degree-of-freedom rotational manipulator.

For general geometric path, it could be defined by a parameter. Specially, arc length ' s ' is used in this paper such that its first and second derivatives present tangential velocity and acceleration. Since the path following problem is with one degree of freedom nature, by choosing the state variables as s and ds/dt, the problem could be converted to s-space, and phase plane technique is usable. Based on the Pontryagin's minimum principle, it is shown that there exists no singular subarc in the minimum time solution [1]. The same result is obtained and used by Bobrow, Dubowsky and Gibson ( 1985 ) [2], and Shin and McKay ( 1985 ) [3], when they develop their construction algorithms for switching curve in phase plane, which are adopted here because they are conceptually easy.

The steady state solution for a circle path is obtained and compared with the maximum constant speed solution. Then, the transient parts are added. From the observations, a control strategy is proposed with a trade-off between various requirements as below :

(1) The speed is as fast as possible with available bounded torque.
(2) The speed is constantwise.
(3) The speed is limited above by available Laser power.
(4) The speed is limited below by the continuity of sintering which does not occur if Laser power is too low.

## Minimum Time Optimal Control Problem




**fig#1.** Scanner System

**A. θx-θy space :** The state variables are chosen as $\theta x$, $\omega_x$, $\theta y$, $\omega_y$.

$$\min J = \min \int_0^{t_f} dt = \min t_f$$

subjected to :
$$d\theta x/dt = \omega_x,$$
$$d\omega_x/dt = g_1(\theta x, \omega_x) + \tau_x,$$
$$d\theta y/dt = \omega_y,$$
$$d\omega_y/dt = g_2(\theta y, \omega_y) + \tau_y,$$

and $-M \leq \tau_x \leq M$, $-M \leq \tau_y \leq M$, where M is a positive value
$S(x,y) = 0$, which is a simple closed curve with no sharp corner.
$x(0) = x_0$, $x(t_f) = x_f$, $v_x(0) = v_{x0}$, $v_x(t_f) = v_{xf}$,
$y(0) = y_0$, $y(t_f) = y_f$, $v_y(0) = v_{y0}$, $v_y(t_f) = v_{yf}$, are given.

with the nonlinear geometric relation between control space and specified path space as shown in fig#1 :
$$\theta x = G1(x, y)$$
$$\theta y = G2(x, y)$$

then $\theta x(0) = \theta x_0$, $\theta x(t_f) = \theta x_f$, $\omega_x(0) = \omega_{x0}$, $\omega_x(t_f) = \omega_{xf}$,
$\theta y(0) = \theta y_0$, $\theta y(t_f) = \theta y_f$, $\omega_y(0) = \omega_{y0}$, $\omega_y(t_f) = \omega_{yf}$,
are some functions of $x_0$, $x_f$, $v_{x0}$, $v_{xf}$, $y_0$, $y_f$, $v_{y0}$, $v_{yf}$.

Note that (G1,G2) is one-to-one mapping between (x,y) and (θx,θy) in the SFF application, and all the derivatives are continuous in working domain [6].

**B. s - space :** $S(x,y) = 0$ could be defined as $x = f_1(s)$ and $y = f_2(s)$, where s is the unit arc length, and the first derivatives of $f_1(s)$ and $f_2(s)$ are continuous since no sharp corner in the path. Let $v = ds/dt$, $u = d^2s/dt^2$,

then $\theta x = G1(f_1(s), f_2(s)) = h_1(s)$

$\theta y = G2(f_1(s), f_2(s)) = h_2(s)$

and $\omega_x = h_1'(s)\, v$, $\quad d\omega_x/dt = h_1''(s)\, v^2 + h_1'(s)\, u$,

$\omega_y = h_2'(s)\, v$, $\quad d\omega_y/dt = h_2''(s)\, v^2 + h_2'(s)\, u$,

Note that $h_1'(s)$ and $h_2'(s)$ are continuous [7].

The minimum time optimal control problem may be converted as

$$\min J = \min \int_0^{t_f} dt = \min t_f$$

subjected to $ds/dt = v$, $\quad dv/dt = u$, with $s(0) = s_0$, $s(t_f) = s_f$, $v(0) = v_0$, $v(t_f) = v_f$,

and

$$-M \le h_1''(s)\, v^2 + h_1'(s)\, u - g_1(h_1(s), h_1'(s)\, v) \le M \quad (C1)$$

$$-M \le h_2''(s)\, v^2 + h_2'(s)\, u - g_2(h_2(s), h_2'(s)\, v) \le M \quad (C2)$$

Based on the Pontryagin's minimum principle, it is shown that there exists no singular subarc in the minimum time solution, the solution is always on one boundary of those inequality constraints.

## Solution of the Minimum Time Optimal Problem

From the Pontryagin's minimum principle, the two-point-boundary-value problem is obtained by the necessary conditions and solved by Shooting method [4] [5]. However, the difficulty is found after the solution of first example is obtained. There exist the second derivatives in those inequality constraints, which are put into the Hamiltonian. Then, the dynamic equations of Lagrangian multipliers contain the third derivatives, $d\lambda/dt = -\partial H/\partial s$, and the forth derivatives appear in the formula of shooting method. The numerical singularity and sensitivity of convergence must be handled. Based on the motivation as below, the phase plane technique is followed [2],[3].

$$\min J = \min \int_0^{t_f} dt = \min \int_{s_0}^{s_f} (1/v)ds$$

In the phase plane, minimizing the cost is conceptually equivalent to maximizing the speed. The basic steps described below are used to construct the optimal trajectory in phase plane.

#1 : To find the feasible region in phase plane, construct $v_{max}(s)$ curve :

Rewrite the inequality constraints, (C1) and (C2)

$L_1(s,v) \le u \le U_1(s,v)$

$L_2(s,v) \le u \le U_2(s,v)$

where

$$L_1(s,v) = [\, -M - h_1''(s)\, v^2 + g_1(h_1(s), h_1'(s)\, v)\,] / h_1'(s)$$

$$L_2(s,v) = [\, -M - h_2''(s)\, v^2 + g_2(h_2(s), h_2'(s)\, v)\,] / h_2'(s)$$

$$U_1(s,v) = [\, M - h_1''(s)\, v^2 + g_1(h_1(s), h_1'(s)\, v)\,] / h_1'(s)$$

$$U_2(s,v) = [\, M - h_2''(s)\, v^2 + g_2(h_2(s), h_2'(s)\, v)\,] / h_2'(s)$$

Given s, $s_0 \le s \le s_f$, find $v_{max}$ such that the acceleration u exists, that is $\max(L_1, L_2) \le \min(U_1, U_2)$

#2 : To perform the forward integration for s, and v, with maximum $u = \min(U_1, U_2)$ or minimum $u = \max(L_1, L_2)$

#3 : To perform the backward integration for s, and v, with maximum $u = \min(U_1, U_2)$ or minimum $u = \max(L_1, L_2)$

#4 : Use #2 and #3 iteratively to construct the trajectory in phase plane such that v(s) is as high as possible without exceeding the $v_{max}(s)$ curve.

#5 : After ( s, v(s) ) curve is obtained, control torques in control space are calculated by

$$\tau_x = h_1''(s)\, v^2 + h_1'(s)\, u - g_1( h_1(s), h_1'(s)\, v )$$
$$\tau_y = h_2''(s)\, v^2 + h_2'(s)\, u - g_2( h_2(s), h_2'(s)\, v )$$

Notice that this solution satisfies not only the state equations and inequality constraints in control space but also the equality constraint in path space. The above steps are implemented by MatLab package ( The MathWorks, Inc., Sherborn, MA ) for the results shown in this paper.

## Example

Model is given by double integral and assume $\theta x = x$, $\theta y = y$,

$$\begin{array}{llll} d\theta x/dt = \omega_x, & \text{with} & \theta x(0) = x(0) = 0, & \theta x(t_f) = x(t_f) = 0, \\ d\omega_x/dt = g_1( \theta x, \omega_x ) + \tau_x, & & \omega_x(0) = v_x(0) = v_{x0}, & \omega_x(t_f) = v_x(t_f) = v_{xf}, \\ d\theta y/dt = v_y, & & \theta y(0) = y(0) = -1, & \theta y(t_f) = y(t_f) = -1, \\ d\omega_y/dt = g_2( \theta y, \omega_y ) + \tau_y, & & \omega_y(0) = v_y(0) = 0, & \omega_y(t_f) = v_y(t_f) = 0, \end{array}$$

and $-M \le \tau_x \le M$, $-M \le \tau_y \le M$, where $M = 1.1715728$

$S(x,y) = 0 = x2 + y2 - r^2$, whole circle, with arc length $2r\pi$, $r = 2$

===> $x = r*\sin(0.5s)$, $y = -r*\cos(s)$, $0 \le s \le 2r\pi$, that is $s_0 = 0$, $s_f = 2r\pi$

Case 1 : $v_{xf} = v_{x0} = ( r*M )^{0.5}$, ==> $v_0 = 1.530733$, $v_f = 1.530733$, results is shown in fig#1.

(a) optimal solution : $t_f* = 8.0$

(b) maximum constant speed solution : $t_f = 8.2094$

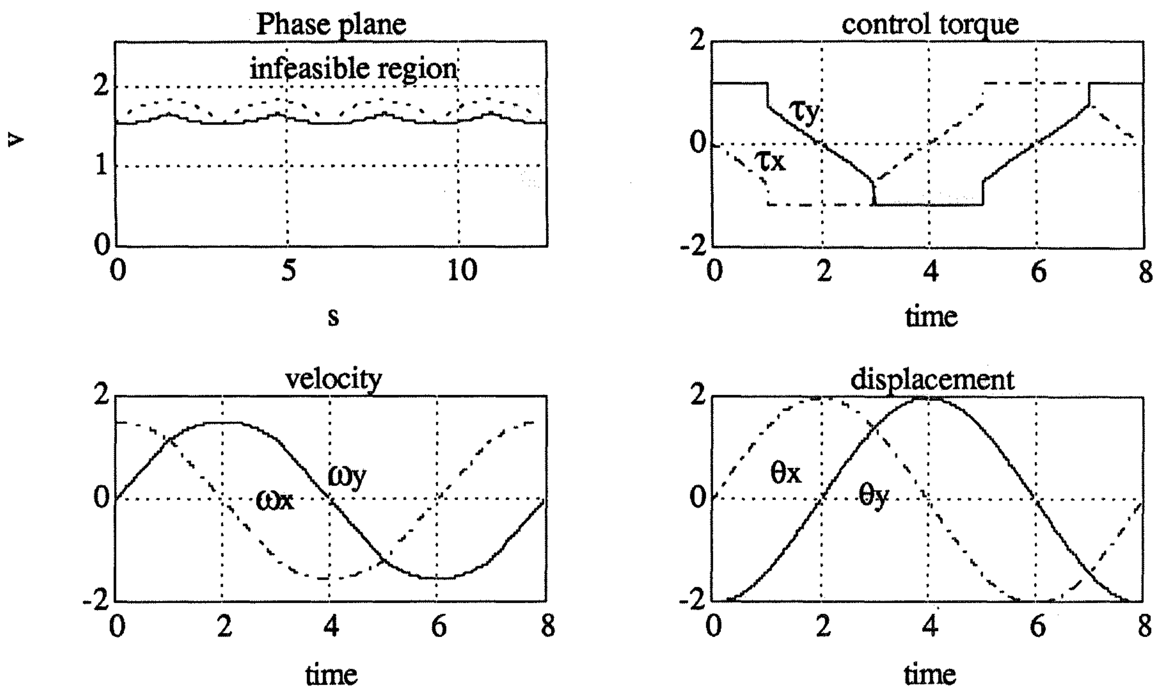
$( t_f - t_f^* ) / t_f = 2.55\%$ which is independent of M and r.

Case 2 : $v_{xf} = v_{x0} = 0$, ==> $v_0 = 0$, $v_f = 0$, results is shown in fig#2.

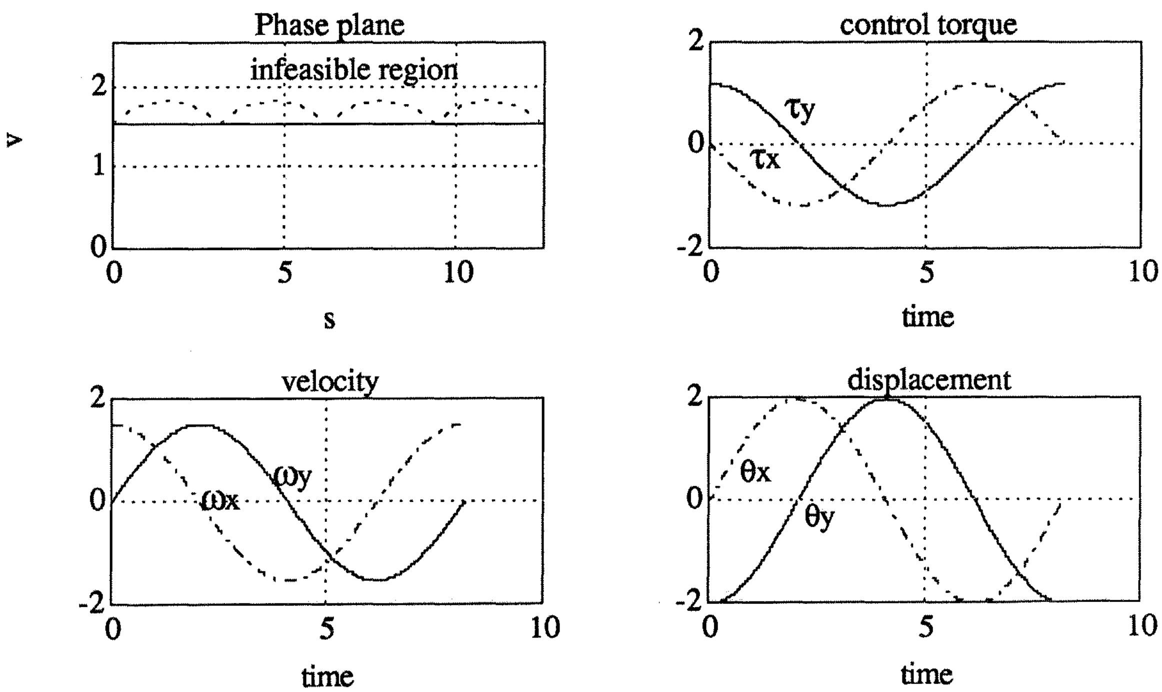
(a) optimal solution : $t_f* = 9.3330$

(b) maximum constant speed solution : $t_f = 9.4993$

$( t_f - t_f^* ) / t_f = 1.75\%$

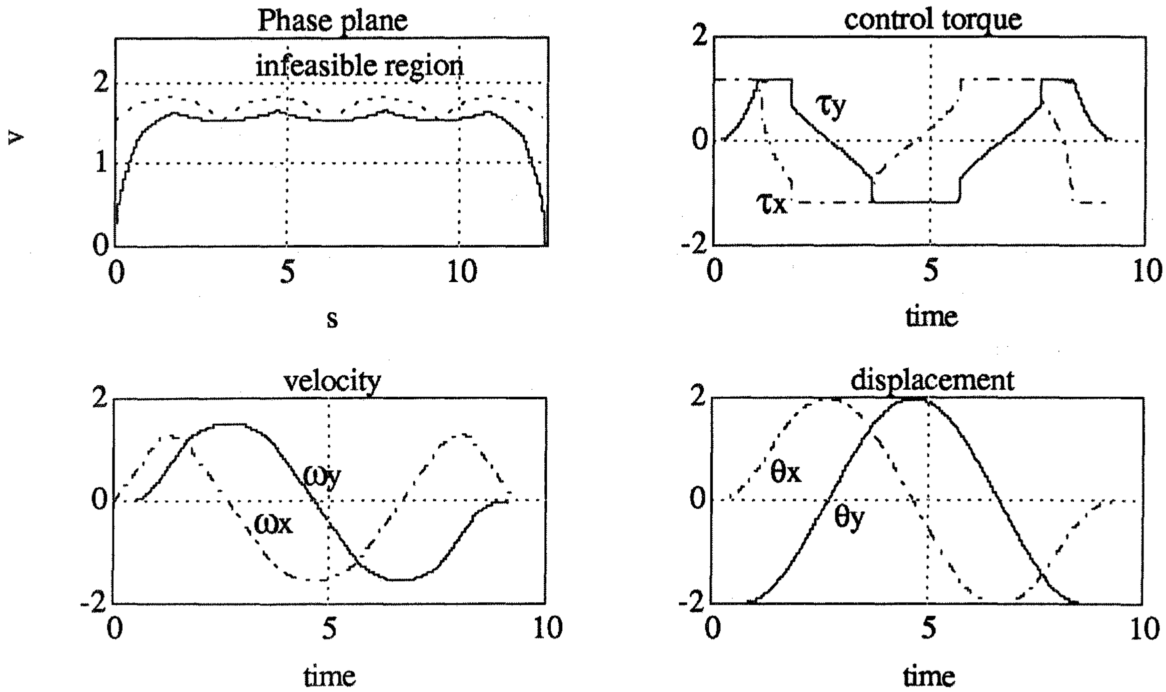



optimal solution : $t_f^* = 8.0$

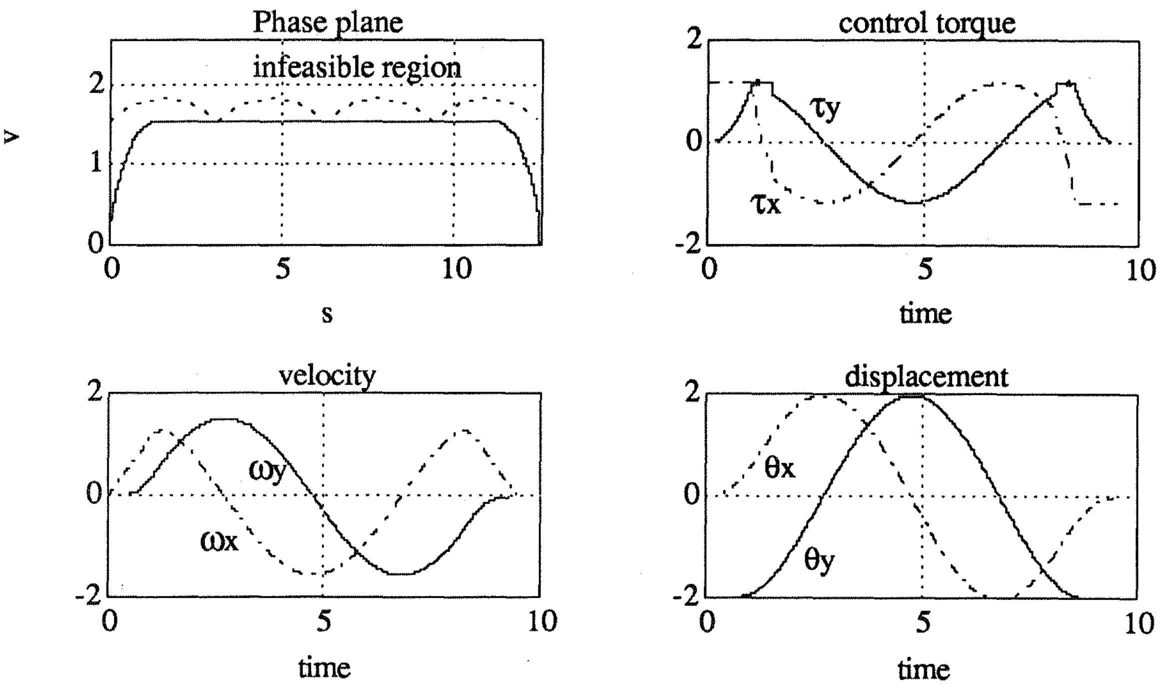



maximum constant speed solution : $t_f = 8.2094$

**fig#2** Case1 : $(t_f - t_f^*) / t_f = 2.55\%$




optimal solution : $t_f^* = 9.3330$

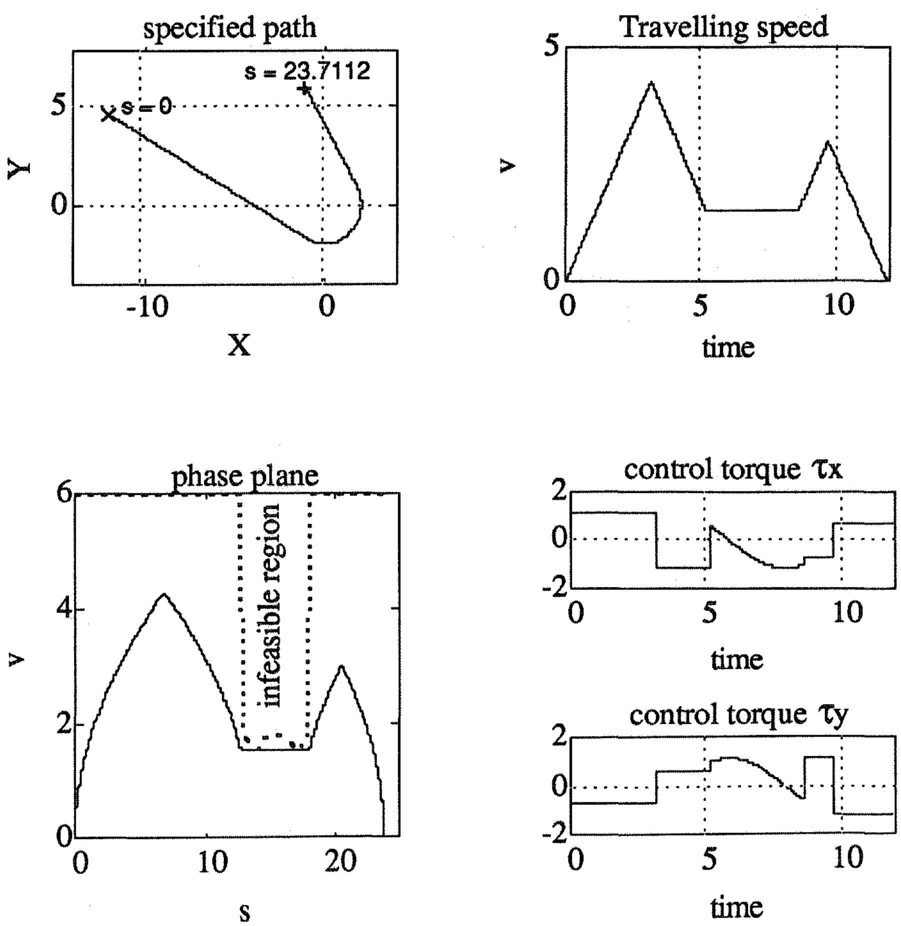



maximum constant speed solution : $t_f = 9.4993$

**fig#3** Case2 : $( t_f - t_f^* ) / t_f = 1.75\%$

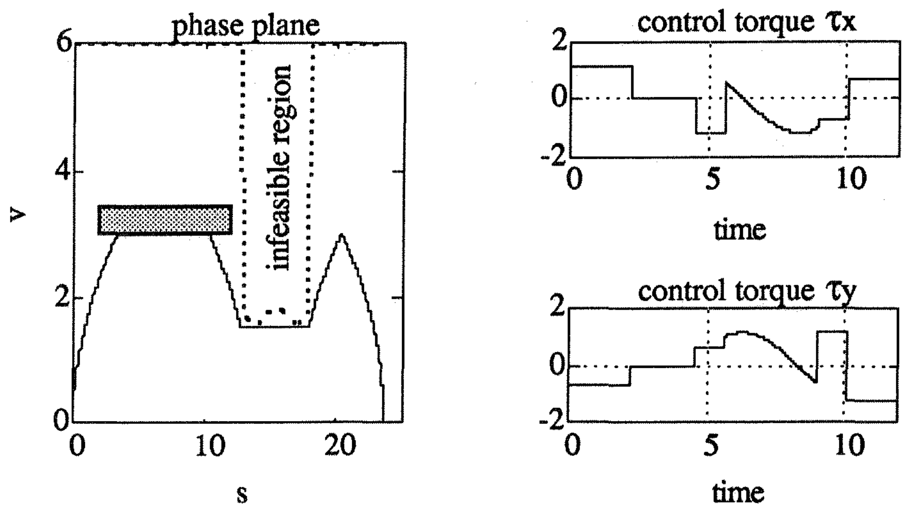
## Control Strategy

In SFF application or other Laser application, one prefers the near constant travelling speed to gain the uniform exposure of Laser power as well as the fastest speed to increase the productivity. From the steady state solution of previous example, the travelling time of the fastest solution is only 2.55% shorter than that of the maximum constant speed solution, which has smooth control torque function. Therefore, if there exists such circlelike arc in the specified path, the maximum constant speed solution is a nice choice. In the example below, there only exists a short part in the travelling path with strict speed limit, a result from the trade-off between smoothness and travelling time is shown in f#4. Since s is chosen to be unit arc length, v(t) presents the travelling speed of Laser spot in the path space. Therefore, if Laser power is on-line adjustable, v(t) is the reference.



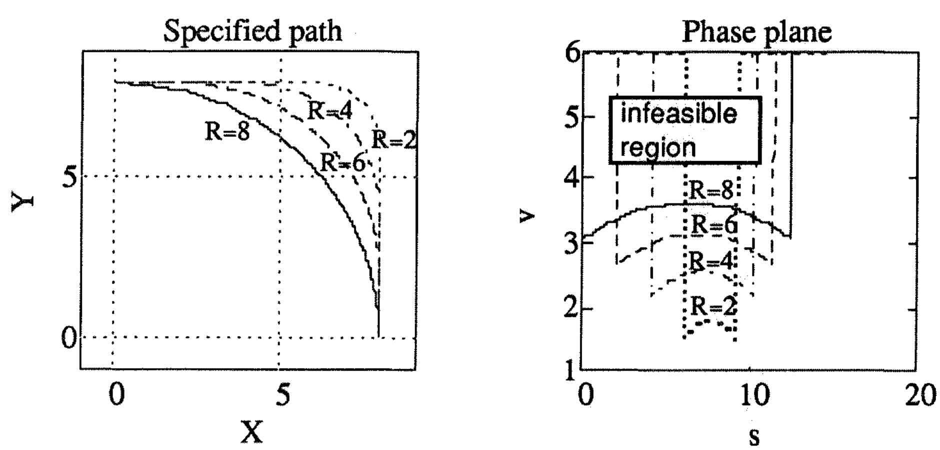

**fig#4** The trade-off between smoothness and travelling time : $t_f = 11.9221$

If the travelling speed is limited above by the available Laser power, it is straight forward to put an extra constraint in phase plane and modify the control trajectory. This is shown in fig#5.




**fig#5** Speed limited by available Laser power : $t_f = 12.3217$

If the corner is too sharp, that is the radius is too small, the speed limit region in phase plane may be very low. Then, the Laser spot must slow down when pass that corner. In order to gain uniform degree of sintering, Laser power need to be turned down. Thus, the speed is limited below by the continuity of sintering which does not occur if Laser power is too low. When, the radius and maximum torque are given, the fastest speed around the corner could be seen in phase plane. The design of the radius at corner should compromise with the speed requirement, available Laser power. The change of infeasible speed region in phase plane corresponding to various radius is shown in fig#6.




**fig#6** Infeasible speed region vs Radius of corner

## Conclusion

Given a scanner system, the maximum torque, the specified path, and other circumstance in SFF application, the question, 'how fast could the scanner be driven ?', is answered in this paper. It is nice to formulate the minimum time optimal control problem in s-space since (1) the problem is one-degree-of-freedom such that the phase-plane technique is useful, (2) v=ds/dt presents the travelling speed along the path which could be referred for any speed constraints when s is unit arc length. The control strategy is proposed. If the path is near circle arc, the infeasible speed region is like that shown in fig#2 and fig#3, the maximum constant speed solution is chosen since it only needs 2.55% more than the time needed by optimal solution, and has smoother control trajectory. If speed is limited by available Laser power, the constraints could be straight forward added into phase plane diagram. If continuity and uniformity of sintering are required, the radius of corner in the path must be restricted. In phase plane diagram, it is easy to see the speed limit around the corner corresponding to the given radius.

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