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Reviewed Paper

Using Mean Curvature of Implicitly Defined Minimal Surface Approximations to Generate New Unit Cells for Lattice Design

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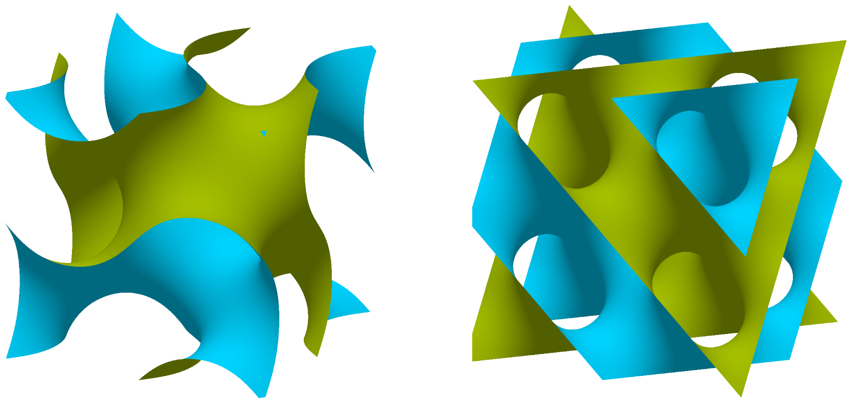
Abstract

Triply Periodic Minimal Surfaces (TPMS) are smoothly varying surfaces that exhibit zero mean curvature at all points on the surface. TPMS can be modeled with high accuracy using discrete differential geometry techniques. However, generating a useful number of unit cells with this approach would be computationally expensive, and variable lattices would be impossible. Level sets of Fourier series approximations are often used instead. While these approximations have continuous geometry, they no longer retain zero mean curvature like the exact TPMS. In this paper, we calculate the mean curvature of the commonly used approximations of the gyroid and D-surface TPMS. Using isosurfaces of the mean curvature from these approximates, we define, similar but unique surface topologies. The development of these surfaces expands the list of lattices available to designers, broadening the lattice design space. Application to other approximations and further study of the application of these new surfaces is discussed.

1 Introduction

There are several equivalent definitions of a *minimal surface* [1]. In this paper, we consider the definition that a minimal surfaces *must have zero mean curvature at all points on the surface*. The mean curvature can be calculated from the average of the two principal curvatures and, for a minimal surface, this mean curvature is always zero. The curvature at all points then must be defined, finite, and the principal curvatures must be equal but opposite. From this property, minimal surfaces have no sharp edges or discontinuities in the surface. Minimal surfaces that are periodic in x, y, and z are known as Triply Periodic Minimal Surfaces (TPMS). The periodicity (like a sine or cosine function) allows the surface to extend and repeat infinitely in 3D space. We can define for each unique TPMS a *unit cell* that represents the smallest element of the topology that can be patterned to produce the surface. Although there are a number of TPMS that have been identified [2, 3], we focus on two common TPMS: the Gyroid and D-surface shown in Fig. 1.

Advances in additive manufacturing (AM) have enabled the economical production of intricate shapes at the “mesoscale”, withing the macro geometry of a component [4]. One type of complex shape that AM enables is periodic 3D structures known as lattice structures [5]. TPMS have seen interest in AM literature as a basis for the design of lattice structures. They are attractive



(a) **(b)**

Figure 1: Images of the (a) Gyroid and (b) D-surface generated using Surface Evolver with blue and green showing the two sides of the surface.

as lattice structures because there are no sharp corners and the introduction of stress concentrations can be reduced or avoided. However, because a mathematical surface has no volume, the infinitely thin surface of the TPMS must be made solid by either "adding" thickness on either side of surface or by filling one side (blue or green in Fig. 1) of the surface resulting in different lattice topologies, effective material properties, failure mechanisms, etc. By applying constant or spatially varying offset(s) to the surfaces before adding material, the amount of material within the lattice can be controlled. When one side of the surface is filled, we refer to the result as a skeletal lattice. The ratio of solid volume to total design volume is commonly used to discuss and compare lattices structures and is known as the Volume Fraction (VF) of the lattice. The design of the lattice's geometry is critically important and implicit design tools (such as nTopology [6]) make it easy to control the topology to elicit a specific response from the lattice, and so various types of structures are necessary to address engineering demands.

Literature has provided approximations for TPMS that allow engineers to readily implement and model these structures in design tools. The Gyroid and D-surface approximations used in this paper are taken originally from Von Schnering and Nesper [3], where they were derived by fitting Fourier series to points on the TPMS surfaces. For convenience we define a shorthand nomenclature for writing sine and cosine functions as:

$$S_{ni} = \sin\left(\frac{2\pi n}{P_i} i\right), \tag{1}$$

$$C_{ni} = \cos\left(\frac{2\pi n}{P_i} i\right), \tag{2}$$

where i is a *spatial coordinate* $\in \{x, y, z\}$, P_i is the period of the lattice in direction i, and n is the number of periods of the trigonometric function in one unit cell (e.g. $1, 2, 3$). When n is unity, it is omitted for simplicity. The approximations of the Gyroid and D-surface can then be written as:

$$C_x S_y + C_y S_z + C_z S_x = t \tag{3}$$

$$S_x S_y S_z + S_x C_y C_z + C_x S_y C_z + C_x C_y S_z = t \tag{4}$$

where t is the level set applied to offset the surface from the zero level set. This nomenclature becomes more useful when writing the mean curvature equations discussed in Section 2. Because these equations represent approximations of TPMS, they no longer inherently respect the properties of a minimal surface, including those that ensure there are no sharp corners. However for the Gyroid and D-surface, the error between the the TPMS and its approximation is small at the zero level set [7].

The existing work done by Von Schnering and Nesper [3] and Gandy et al. [7] to generate the approximations and characterize them was performed at a single constant level set that best fit the approximation to the TPMS. However, in application, that same level set value is rarely if ever used to generate lattices. The mean curvature of any approximations to TPMS have not been described as a function of level set to the best of the authors' knowledge, and the actual surface curvature of the lattice structures being used is unknown. Because the mean curvature has implications on the performance of the lattice, characterizing this metric is highly relevant. Große-Brauckmann [8] does discuss methods for generating offset gyroid companion surfaces that exhibit constant (or near constant) mean curvature using Surface Evolver [9]. However, for use in engineering applications, generation of these surfaces throughout the design volume is not feasible with this technique as convenient approximations of these surfaces were not presented. Li et al. [10] generated a piecewise modification of the Gyroid that maintained the connectivity of the structure at large level set values, allowing connected Gyroid lattices with lower VF to be generated. The approach was not extended to other surfaces.

In this paper, we explore the analytical derivation of mean curvature for the aforementioned approximations to the Gyroid and D-surface at non-zero level set values with a general method that can be applied to any equation-based lattice. We explore defining new lattices using the analytically determined mean curvature of existing TPMS approximations to study the resulting surface's curvature, topology, and geometry relative to their original approximation.

2 Methods

The surfaces modeled using Eqs. 3 and 4 are implicitly defined functions (i.e. $f(x,y,z)$). The curvature of an implicit field $f(x,y,z)$ can be calculated using the divergence of the unit normal [11]:

$$H = -\frac{1}{2}\nabla \cdot \left(\frac{\nabla F}{|\nabla F|}\right) \tag{5}$$

In our exploratory study, we make the simplifying assumption that the periodicity in each unit cell is equal and constant $(P_x = P_y = P_z = 1)$. This allows for the derived curvature equation to be simplified and easily interpreted even if they are only accurate for cubic unit cells of constant size. These assumptions are only for presenting the results — the method works without these simplifications.

Using the MATLAB symbolic toolbox, we compute the mean curvature of the implicit TPMS approximations. The mean curvature equations were algebraically simplified and are presented in appendix A. We visualize the 3D curvature field by plotting the Gyroid (Fig. 2) and D-surface (Fig. 3) approximations at fifteen level set values ranging from 5% of the fields minimum value, to 95% of the fields maximum value, overlaying a color map of the surface's curvature. This allows us to visualize what the curvature of the surfaces of lattice structures are across the range of volume fractions. The level sets of zero in Figs. 2 and 3 are the original fits of the TPMS. The larger the magnitude of the level set, the larger the observed variation in mean curvature across the surface. For example, the Gyroid in Fig. 2 shows non-constant mean curvature at level sets far from zero; the D-surface has similar behavior. This behaviour is not unexpected, as the Fourier fit to the TPMS was at only the zero level set value and any deviations from this level set are inherently less accurate.

The mean curvature of the surface approximations can be used to generate implicit fields, and by taking isosurfaces (aka level sets) of the mean curvature field, we generate new variants on the Gyroid and D-surface TPMS. We refer to these new surfaces (and the lattices generated with them) as the Mean Curvature Gyroid (MC Gyroid) and Mean Curvature D-Surface (MC D-surface) to differentiate them from the existing Gyroid and D-surface approximations. Appendix B contains code to calculate the mean curvature of both the TPMS approximations and the MC surfaces symbolically. The mean curvature equations were also implemented in nTopology for comparison to the original approximations and to enable efficient lattice modeling. These nTopology notebooks are made available at https://github.com/jwf23/Mean-Curvature-Surfaces and can be imported as custom blocks to utilize the MC lattice topologies.

By computing the mean curvature defined in Eq. 5, applied to the MC Gyroid and MC D-surface equations (Eqs. A.1 and A.2), the mean curvature of the MC Gyroid and MC D-surface is determined. We visualize the MC Gyroid and MC D-surface colored with their respective curvatures in Figs. 4 and 5.

3 Results & Discussion

The skeletal lattices of the MC and traditional variants of the two surfaces where compared at the same volume fractions to assess the level of difference within the lattices produced (Table 1). The two models were aligned to the same reference frame as the unit cell, and the MC variant was subtracted from the traditional. The percent difference was computed by dividing the volume of the subtracted model by the volume of (either equal volume) unit cell. This represents the percentage of the MC Gyroid or MC D-surface that is *not shared* by the their traditional variants.

Table 1: Computed volumetric differences between the MC and traditional Gyroid and D-surface skeletal lattices at different volume fractions.

Gyroid		D-surface	
VF	% Difference	VF	% Difference
50%	1.5%	50%	2.1%
25%	5.2%	25%	5.3%
10%	12%	10%	11%
5%	17%	8.5%	15%

We present a visual representation of row three of Table 1 in Figs. 6 and 7 to illustrate the similar but unique geometry of the lattices. The skeletal lattices of the two variants of the Gyroid

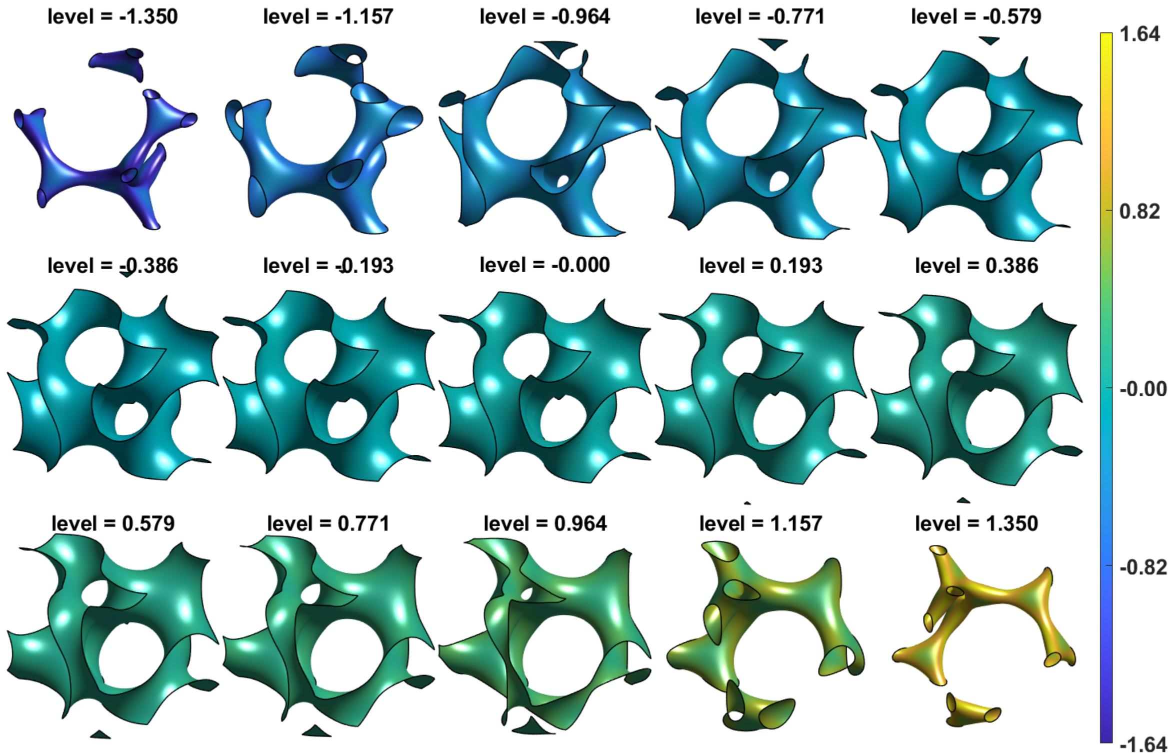


Figure 2: Gyroid mean curvature across a range of level sets showing existing nonzero mean curvature, especially when deviating from the zero level set.

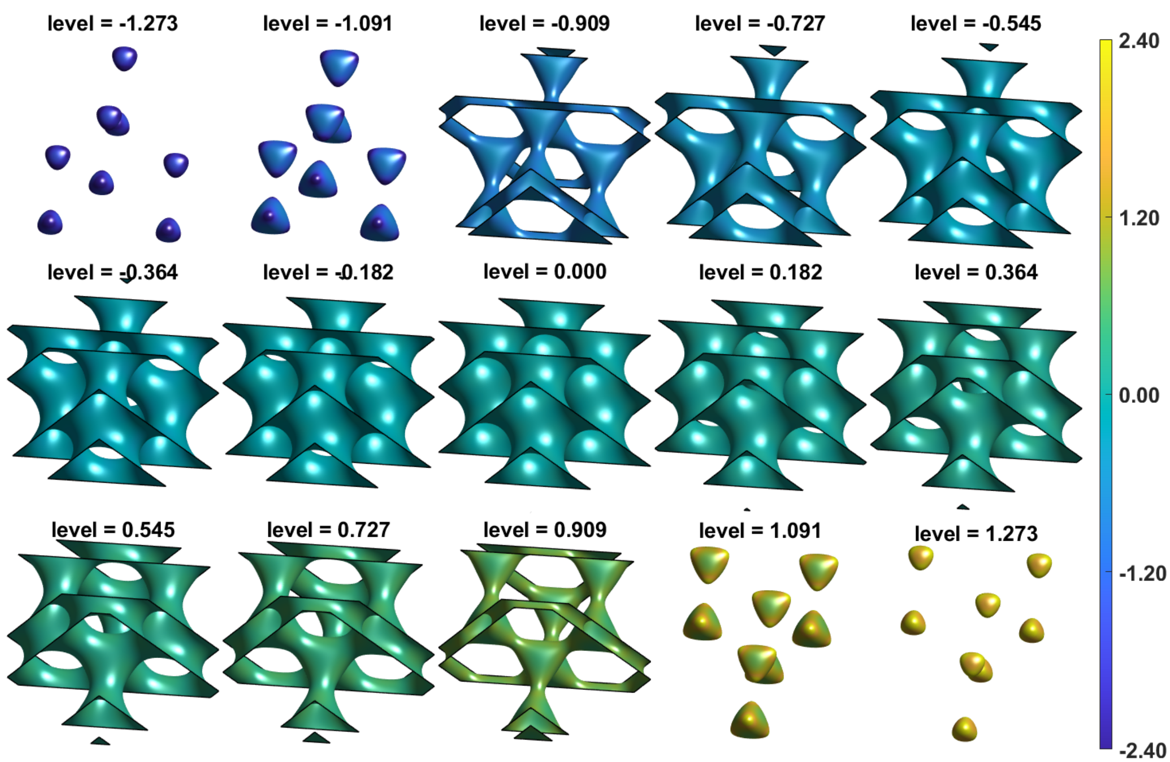


Figure 3: D-surface Mean curvature across a range of level sets showing existing nonzero mean curvature. The limits of the color bar were fixed to match Fig. 5 for a more direct comparison and for greater contrast when $|\text{level}| \leq 0.909$. The minimum and maximum mean curvature values are ± 5.05.

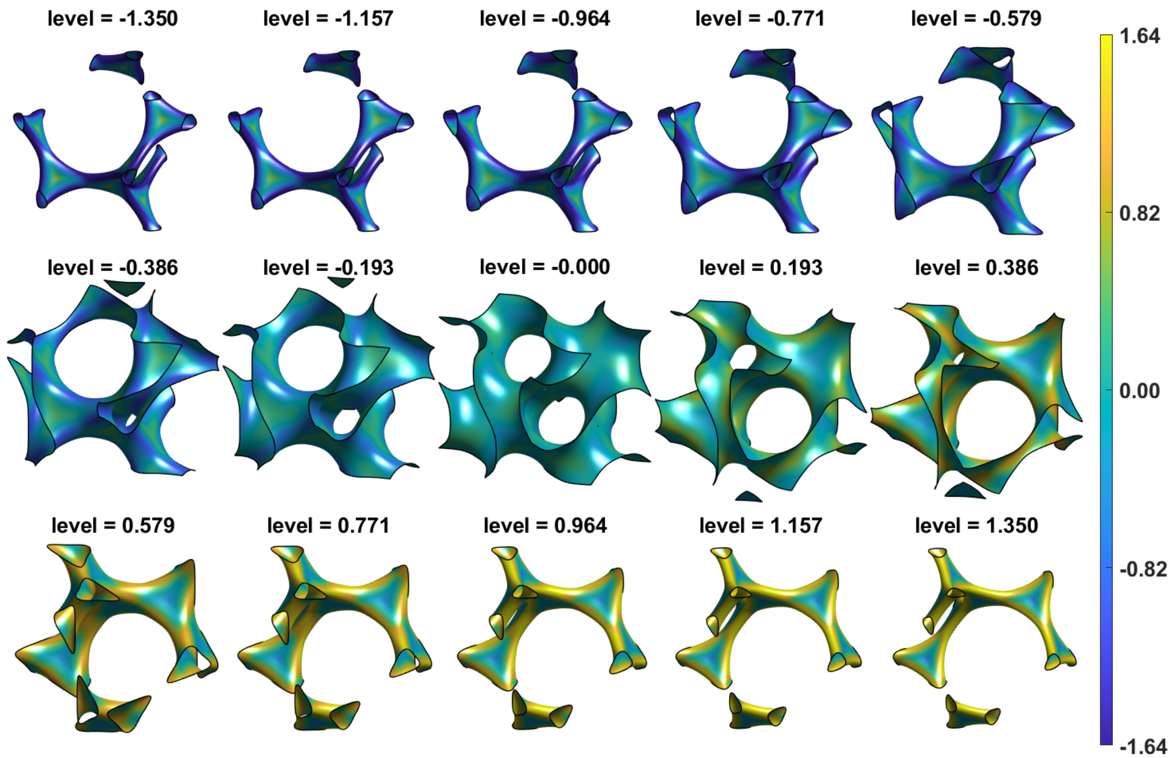


Figure 4: Level-set sweep of the surface defined by the mean curvature of the Gyroid. The limits of the color bar were fixed to match Fig. 2 for more direct comparison. The minimum and maximum mean curvature values are ± 3.03.

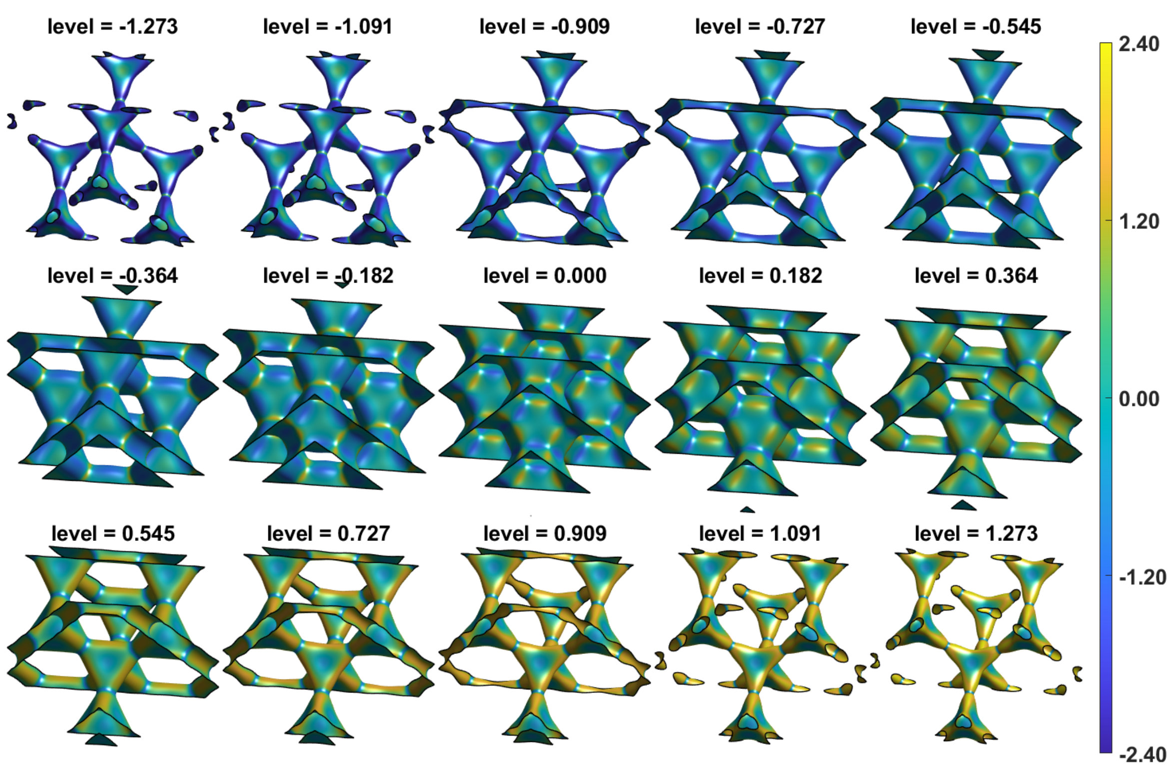
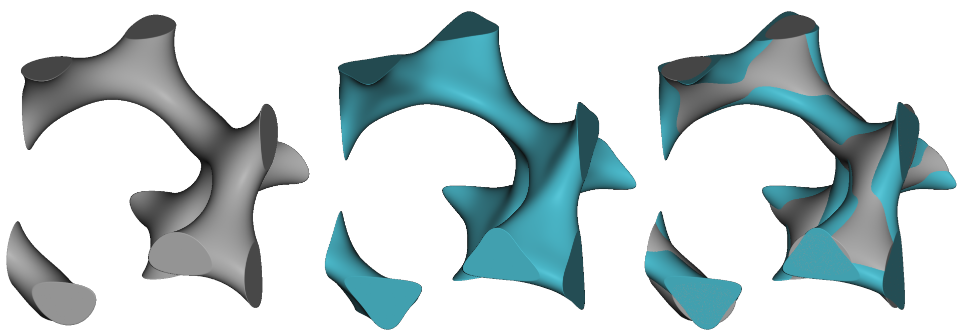


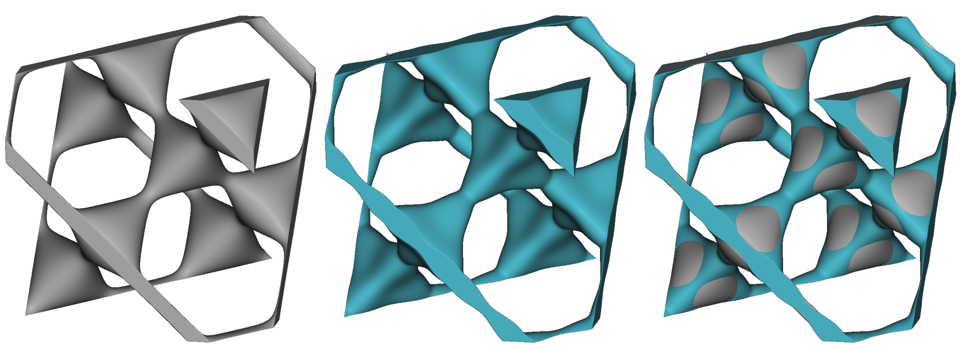
Figure 5: Level-set sweep of the surface defined by the mean curvature of the D-surface.

and D-surface are shown separately as well as overlaid. Regions that appear blue in Figs. 6c and 7c exist in the MC surface and those that are gray exist in the original surface. This mass redistribution contributes to the skeletal lattices of the MC variants reaching lower volume fractions before the lattice becomes discontinuous, and when the other side of the surface is filled, higher volume fractions can be reached by the MC variants before internal voids form. The traditional gyroid skeletal lattice becomes discontinuous at a VF of 1.9% (level set of 1.41), where the MC gyroid skeletal lattice remains continuous until a VF of 0.04% at a level set of 12. We note that for the MC Gyroid, at low VF (high level set values) the "thin" regions of the lattice do exhibit a high aspect ratio cross section, which could impact manufacturability. Similarly, we can look at the D-surface which becomes discontinuous at a VF of 8% (level set of 1.76). Compared to the MC D-surface which is continuous until a VF of 0.6% at a level set of 3.8. The extension of the range of volume fractions is comparable to those demonstrated by Li et al. [10] for the Gyroid. However, where Li et al. used a piecewise function to maintain the same geometry as the Gyroid generated by Equation 3, the MC variants are new surfaces that share the same general topology, but unique geometry.



(a) **(b)** **(c)**

Figure 6: Comparison of geometry at 10% volume fraction showing (a) the Gyroid, (b) the MC Gyroid, and (c) the Gyroid and MC Gyroid overlaid.



(a) **(b)** **(c)**

Figure 7: Comparison of geometry at 10% volume fraction showing (a) the D-surface, (b) the MC D-surface, and (c) the D-surface and MC D-surfaces overlaid.

The MC D-surface exhibits more pronounced "necking" compared to the traditional D-surface (see Figure 7). The impact that this has on the properties of the lattice have not yet been explored, but has the potential to produce structures with more consistent failure mechanisms or tuned failure criteria.

4 Conclusion & Future Work

We have defined two new surfaces using an implicit field generated by analytically determining the mean curvature of approximations of two commonly used TPMS – the Gyroid and D-surface. We computed the mean curvature of these two new surfaces which we define as the MC Gyroid and MC D-surface. The geometric differences between skeletal lattices generated with the original and MC equations were quantified at equal volume fractions. Although similar in topology, the MC variants have different material distributions, and the percent difference increases as the level set moves away from zero. In addition, the MC variants remain connected for larger ranges of volume fractions than the original surfaces. The value of these new lattices for engineering applications has yet to be explored, although the geometric differences between the new surfaces and the original could prove valuable.

We have demonstrated the theory behind generating new surfaces using this approach, as well as the means to implement these new surfaces in design software. The performance of these lattices needs to be evaluated to understand their applicability to different engineering applications, but is beyond the scope of this work. Numerical and experimental methods should be applied to study the mechanical response of the lattices generated using the new MC variants. Our methods employed here to generate new surfaces from the mean curvature should be applied to other TPMS approximations beyond the Gyroid and D-surface. In addition, other pathways, along with mean curvature, should be considered to produce additional new surfaces for lattice generation, such as the Gaussian curvature and the derivatives of the field. The greater variety of lattice structures available will expand design freedom and allow for more informed design decisions.

References

[1] William Meeks III and Joaquín Pérez. "The classical theory of minimal surfaces". In: *Bulletin of the American Mathematical Society* 48.3 (2011), pp. 325–407.

[2] Alan Hugh Schoen. *Infinite periodic minimal surfaces without self-intersections*. National Aeronautics and Space Administration, 1970.

[3] HG Von Schnering and R Nesper. "Nodal surfaces of Fourier series: fundamental invariants of structured matter". In: *Zeitschrift für Physik B Condensed Matter* 83.3 (1991), pp. 407–412.

[4] Chen Zhang et al. "Design for laser powder bed additive manufacturing of AlSi12 periodic mesoscale lattice structures". In: *The International Journal of Advanced Manufacturing Technology* 113.11 (2021), pp. 3599–3612.

[5] Dhruv Bhate. "Four questions in cellular material design". In: *Materials* 12.7 (2019), p. 1060.

[6] nTopology. *v2.28*. `https://ntopology.com/`. 2021.

[7] Paul JF Gandy et al. "Nodal surface approximations to the P, G, D and I-WP triply periodic minimal surfaces". In: *Chemical physics letters* 336.3-4 (2001), pp. 187–195.

[8] Karsten Große-Brauckmann. "Gyroids of constant mean curvature". In: *Experimental Mathematics* 6.1 (1997), pp. 33–50. DOI: `em/1047565282`.

[9] Kenneth A Brakke. "The Surface Evolver". In: *Experimental Mathematics* 1.2 (1992), pp. 141–165. DOI: `10.1080/10586458.1992.10504253`.

[10] Dawei Li et al. “Optimal design and modeling of gyroid-based functionally graded cellular structures for additive manufacturing”. In: *Computer-Aided Design* 104 (2018), pp. 87–99.
[11] Ron Goldman. “Curvature formulas for implicit curves and surfaces”. In: *Computer Aided Geometric Design* 22.7 (2005), pp. 632–658.

A mean curvature equations for the Gyroid and D-surface

Mean curvature of the Gyroid approximation given in (Eq. 3):

$$
\begin{aligned}
-\Big(& C_x^3 S_y - 2C_z S_x - 2C_y S_z - 2C_x S_y + C_z^3 S_x + C_y^3 S_z \\
& + 2C_x C_y^2 S_y + 2C_x^2 C_z S_x + 3C_x C_z^2 S_y + 3C_x^2 C_y S_z + 3C_y^2 C_z S_x \\
& + 2C_y C_z^2 S_z - 3C_x^2 C_y^3 S_z - 3C_x^3 C_z^2 S_y - 3C_y^2 C_z^3 S_x \\
& - 3C_x^2 C_y^2 C_z S_x - 3C_x C_y^2 C_z^2 S_y - 3C_x^2 C_y C_z^2 S_z + 6C_x C_y C_z S_x S_y S_z \Big) \\
& \Big/ \left(2\left((C_x C_z - S_x S_y)^2 + (C_x C_y - S_y S_z)^2 + (C_y C_z - S_x S_z)^2 \right)^{3/2} \right) = t
\end{aligned}
\tag{A.1}
$$

Mean curvature of the D-surface approximation (given in Eq. 4):

$$
\begin{aligned}
\Big(\sqrt{2} \Big(& 3C_x C_y S_z + 3C_x C_z S_y + 3C_y C_z S_x + 3S_x S_y S_z \\
& - C_{3x} C_{3y} S_z - C_{3x} C_{3z} S_y + C_{3x} S_{3y} C_z + C_{3x} S_{3z} C_y \\
& - C_{3y} C_{3z} S_x + C_{3y} S_{3x} C_z + C_{3y} S_{3z} C_x + C_{3z} S_{3x} C_y \\
& + C_{3z} S_{3y} C_x - S_{3x} S_{3y} S_z - S_{3x} S_{3z} S_y - S_{3y} S_{3z} S_x \Big) \Big) \\
& \Big/ \left(2\left(3C_{2x} C_{2y} C_{2z} - S_{2x} S_{2z} - S_{2y} S_{2z} - S_{2x} S_{2y} + 3 \right)^{3/2} \right) = t
\end{aligned}
\tag{A.2}
$$

B MATLAB code for generating mean curvature

This appendix contains the code used to generate equations for the mean curvature of the traditional surfaces as well as the MC surfaces. The equations are converted into MATLAB functions prior to evaluation for improved performance.

Code 1: MATLAB Code for generating the MC equations and their curvature

```
% DEFINE A SYMBOLIC FUNCTION
syms f(x, y, z)
% KEY IN THE EQUATION OF THE LATTICE
% Gyroid
f(x, y, z) = cos(x).*sin(y) + cos(y).*sin(z) + cos(z).*sin(x);
% D-surface
%f(x, y, z) = sin(x).*sin(y).*sin(z) + sin(x).*cos(y).*cos(z) + cos(x).*sin(y).*cos(z) + cos
    (x).*cos(y).*sin(z);

% TAKE A DERIVATIVE WRT EACH INDEPENDENT VARIABLE
Dfx = diff(f, x); Dfy = diff(f, y); Dfz = diff(f, z);
% GET THE LENGTH OF THE GRADIENT
Dflen = sqrt(Dfx^2 + Dfy^2 + Dfz^2);
% COMPUTE THE MEAN CURVATURE BY DIVERGENCE DEFINITION
meancrv = -(1/2)*(diff(Dfx / Dflen, x) + diff(Dfy / Dflen, y) + diff(Dfz / Dflen, z));

% REPEAT PROCESS FOR MC SURFACE
clear Dfx Dfy Dfz Dfx Dflen
% TAKE A DERIVATIVE WRT EACH INDEPENDENT VARIABLE
Dfx = diff(meancrv, x); Dfy = diff(meancrv, y); Dfz = diff(meancrv, z);
% GET THE LENGTH OF THE GRADIENT
Dflen = sqrt(Dfx^2 + Dfy^2 + Dfz^2);
% COMPUTE THE MEAN CURVATURE BY DIVERGENCE DEFINITION
meancrvofMC = - (1/2)*(diff(Dfx / Dflen, x) + diff(Dfy / Dflen, y) + diff(Dfz / Dflen, z));

% CONVERT SYMBOLIC EQUATIONS INTO MATLAB FUNCS FOR FAST EVALUATION
cFH = matlabFunction(meancrvofMC);
mFH = matlabFunction(meancrv);
fFH = matlabFunction(f);

% SIMPLIFY THE MC EQUATION TO COMBINE TERMS FOR VIEWING
meancrv = simplify(meancrv);
```